STUDY GUIDE OF *CALCULUS I*

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Abstract. The three main subjects in *Calculus I* are **limits**, **differentiation**, and **integration**. This Study Guide covers the key topics and problems featured in the Tests and the Final.

(I). This Guide presents a collection of problems organized by topic. These problems closely reflect those found on the Tests and the Final in terms of type, difficulty, and wording.

(II). Complete and understand all problems **independently**. Solutions are provided, but use them **only** to verify answers and enhance understanding.

Contents

Topic 1. Limits

Topic 1.A. Limits.

Definition (Limits).

- $$\lim_{x\to a} f(x) = L,$$
  if $f(x)$ approaches a number L as x approaches a.
- $$\lim_{x\to a-} f(x) = L,$$
  if $f(x)$ approaches a number L as x approaches a from the left.
- $$\lim_{x\to a+} f(x) = L,$$
  if $f(x)$ approaches a number L as x approaches a from the right.

Theorem (The squeeze theorem)**.** *If*

$$f(x) \le g(x) \le h(x) \quad \text{and} \quad \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

Remark.

- $$1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \text{as } x \to 0.$$
- $$-1 \le \sin x, \cos x \le 1 \quad \text{for all } x.$$

Problem 1. *Find the limits.*

(a).

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}.$$

(b).

$$\lim_{x\to 2} \frac{4-x^2}{x^2-3x+2}.$$

(c).

$$\lim_{x\to 0} \frac{\tan x}{x}.$$

(d).

$$\lim_{x\to 0} \frac{\sin(3x)}{x}.$$

Answer.

(a).

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} &= \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x})^2-1^2}\\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}\\ &= \lim_{x\to 1}\sqrt{x}+1\\ &= \sqrt{1}+1\\ &= 2.\end{aligned}$$

(b).

$$\begin{aligned}\lim_{x\to 2}\frac{4-x^2}{x^2-3x+2} &= \lim_{x\to 2}\frac{(2-x)(2+x)}{(x-1)(x-2)}\\ &= \lim_{x\to 2}\frac{-(x-2)(2+x)}{(x-1)(x-2)}\\ &= \lim_{x\to 2}\frac{-(2+x)}{x-1}\\ &= \frac{-(2+2)}{2-1}\\ &= -4.\end{aligned}$$

(c). As $x\to 0$,

$$1-\frac{x^2}{6}\le\frac{\sin x}{x}\le 1.$$

Since

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)=1 \quad\text{and}\quad \lim_{x\to 0}1=1,$$

by the squeeze theorem,

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Then

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x\cdot\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{\cos 0}=1.$$

(d). As $y\to 0$,

$$1-\frac{y^2}{6}\le\frac{\sin y}{y}\le 1.$$

Since

$$\lim_{y\to 0}1-\frac{y^2}{6}=1 \quad\text{and}\quad \lim_{y\to 0}1=1,$$

by the squeeze theorem,

$$\lim_{y\to 0}\frac{\sin y}{y}=1.$$

By a change of variables $y = 3x$,

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = 3\lim_{x\to 0} \frac{\sin(3x)}{3x} = 3\lim_{y\to 0} \frac{\sin y}{y} = 3.$$

Topic 1.B. Limits involving infinity.

Theorem (Limits involving infinity)**.**

(i).

$$\frac{\textit{number}}{0} = \begin{cases} \infty & \textit{if positive,} \\ -\infty & \textit{if negative.} \end{cases}$$

Use test points near the base point to determine the sign.

(ii).

$$\frac{\textit{number}}{\pm\infty} = 0.$$

(iii). The limit of the quotient of two polynomials at infinity is the same as the quotient of the leading terms. For example,

$$\lim_{x\to\infty} \frac{18x^3 + 5x^2 - 2x}{42x^3 + x^2 + 3x + 2} = \lim_{x\to\infty} \frac{18x^3}{42x^3} = \frac{18}{42} = \frac{3}{7}.$$

Problem 2. *Find the limits.*

(a).

$$\lim_{x\to 3-} \frac{x}{9 - x^2}, \quad \lim_{x\to 3+} \frac{x}{9 - x^2}, \quad \lim_{x\to 3} \frac{x}{9 - x^2}.$$

(b).

$$\lim_{x\to -2-} \frac{x - 1}{x^2 + 4x + 4}, \quad \lim_{x\to -2+} \frac{x - 1}{x^2 + 4x + 4}, \quad \lim_{x\to -2} \frac{x - 1}{x^2 + 4x + 4}.$$

(c).

$$\lim_{x\to\infty} \frac{2x + 3}{\sqrt{x^2 + 1}}.$$

(d).

$$\lim_{x\to\infty} \frac{\sin(6x)}{13x}.$$

Answer.

(a).

$$\frac{x}{9 - x^2} = \frac{x}{(3 - x)(3 + x)}.$$

Test at $x = 2.9$ for $x \to 3-$:

$$\frac{2.9}{(3 - 2.9)(3 + 2.9)} > 0. \quad \Rightarrow \quad \lim_{x\to 3-} \frac{x}{(3 - x)(3 + x)} = \infty.$$

Test at $x = 3.1$ for $x \to 3+$:

$$\frac{3.1}{(3 - 3.1)(3 + 3.1)} < 0. \quad \Rightarrow \quad \lim_{x\to 3+} \frac{x}{(3 - x)(3 + x)} = -\infty.$$

Hence,

$$\lim_{x\to 3} \frac{x}{9 - x^2} \text{ does not exist and is not } \infty \text{ or } -\infty.$$

(b).

$$\frac{x-1}{x^2+4x+4} = \frac{x-1}{(x+2)^2}.$$

Test at $x = -2.1$ for $x \to -2-$:

$$\frac{-2.1-1}{(-2.1+2)^2} < 0. \quad \Rightarrow \quad \lim_{x\to -2-} \frac{x-1}{(x+2)^2} = -\infty.$$

Test at $x = -1.9$ for $x \to -2+$:

$$\frac{-1.9-1}{(-1.9+2)^2} < 0. \quad \Rightarrow \quad \lim_{x\to -2+} \frac{x-1}{(x+2)^2} = -\infty.$$

Hence,

$$\lim_{x\to -2} \frac{x-1}{x^2+4x+4} = -\infty.$$

(c).

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{2x}{x} = 2.$$

(d). Because $-1 \le \sin(6x) \le 1$,

$$-\frac{1}{13x} \le \frac{\sin(6x)}{13x} \le \frac{1}{13x} \quad \text{as } x \to \infty.$$

Since

$$\lim_{x\to\infty} -\frac{1}{13x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{13x} = 0,$$

by the squeeze theorem,

$$\lim_{x\to\infty} \frac{\sin(6x)}{13x} = 0.$$

Topic 1.C. Continuity.

Definition (Continuity). We say that f is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

Problem 3. *Let*

$$f(x) = \begin{cases} \frac{\sqrt{x}-3}{x-9}, & x \neq 9, \\ A, & x = 9. \end{cases}$$

Find A such that $f(x)$ is continuous.

Answer. Compute that

$$\begin{aligned} \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{(\sqrt{x})^2 - 3^2}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \\ &= \frac{1}{\sqrt{9}+3} \end{aligned}$$

$$= \frac{1}{6}.$$

The function f is continuous if

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = f(9) = A,$$

that is,

$$A = \frac{1}{6}.$$

Topic 2. Differentiation

Topic 2.A. Derivatives by limits.

Definition (Derivative by limit).

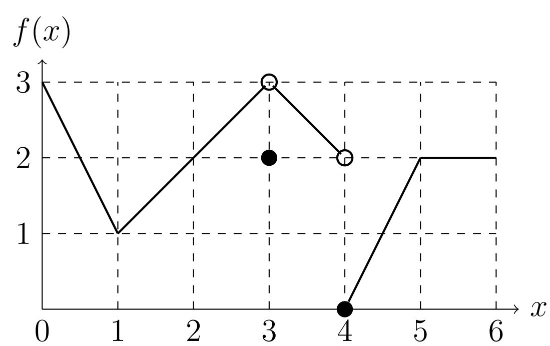
- The derivative of a function f at a point x is
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$
- The tangent to f at the point (x_0, y_0) has slope $f'(x_0) = m$ and equation
$$y - y_0 = m\,(x - x_0)\,.$$

Theorem.

(i). If f is continuous, then the graph of f has no breaks or jumps. (However, f may have corners.)

(ii). If f is differentiable, then f is continuous and the graph of f has no corners.

Problem 4. *Let f be defined as follows.*



(a). List any x values where f is not continuous.

(b). List any x values where f is not differentiable.

(c). Sketch the graph of $f'(x)$.

Answer.

(a). The function f is not continuous at $x = 3, 4$.

(b). The function f is not differentiable at $x = 1, 3, 4, 5$.

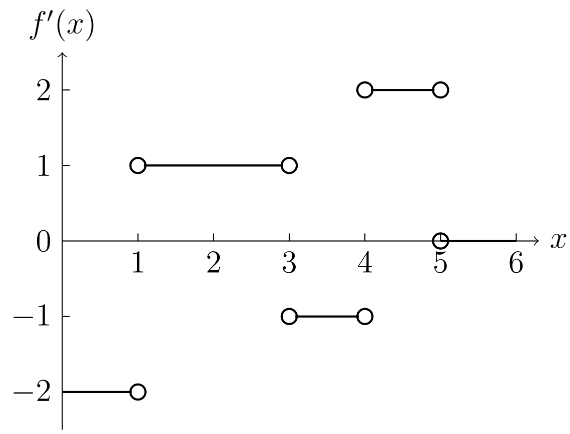
(c).

$$\text{The slope is } \begin{cases} \text{from } (0,3) \text{ to } (1,1): & \frac{1-3}{1-0} = -2, \\ \text{from } (1,1) \text{ to } (3,3): & \frac{3-1}{3-1} = 1, \\ \text{from } (3,3) \text{ to } (4,2): & \frac{2-3}{4-3} = -1, \\ \text{from } (4,0) \text{ to } (5,2): & \frac{2-0}{5-4} = 2, \\ \text{from } (5,2) \text{ to } (6,2): & \frac{2-2}{6-5} = 0. \end{cases}$$

That is, the derivative

$$f'(x) = \begin{cases} -2 & \text{on } (0,1), \\ 1 & \text{on } (1,3), \\ -1 & \text{on } (3,4), \\ 2 & \text{on } (4,5), \\ 0 & \text{on } (5,6). \end{cases}$$

Hence, $f'(x)$ is as follows.



Problem 5. *Let*

$$f(x) = \sqrt{2x+1}.$$

(a). Find $f'(x)$ using the limit definition of derivative.
(b). Find the equation of the tangent to $y = f(x)$ at $(4, f(4))$.

Answer.
(a).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{2x+2h+1} - \sqrt{2x+1}\right)\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{2x+2h+1}\right)^2 - \left(\sqrt{2x+1}\right)^2}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(2x+2h+1)-(2x+1)}{h\left(\sqrt{2x+2h+1}+\sqrt{2x+1}\right)}\\
&= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h+1}+\sqrt{2x+1}\right)}\\
&= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}\\
&= \frac{2}{\sqrt{2x+1}+\sqrt{2x+1}}\\
&= \frac{2}{2\sqrt{2x+1}}\\
&= \frac{1}{\sqrt{2x+1}}.
\end{aligned}$$

(b). Since

$$f'(4) = \frac{1}{\sqrt{2\cdot 4+1}} = \frac{1}{3} \quad \text{and} \quad f(4) = \sqrt{2\cdot 4+1} = 3,$$

the equation of the tangent at $(4, f(4))$ is

$$y - 3 = \frac{1}{3}(x-4),$$

that is,

$$y = \frac{1}{3}x + \frac{5}{3}.$$

Topic 2.B. Derivatives by rules.

Theorem (Derivative rules)**.**

(i). Derivatives of powers:

$$(x^n)' = nx^{n-1}.$$

In particular,

$$\left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

(ii). Derivatives of trigonometric functions:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x.$$

$$(\csc x)' = -\csc x\cot x, \quad (\sec x)' = \sec x\tan x, \quad (\cot x)' = -\csc^2 x.$$

(iii). Derivatives of exponential and logarithmic functions:

$$(e^x)' = e^x \quad \textit{and} \quad (\ln x)' = \frac{1}{x}.$$

(iv). The product rule:

$$(fg)' = f'g + fg'.$$

(v). The quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

(vi). The chain rule:

$$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right)\cdot g'(x).$$

Problem 6. *Find the derivatives.*

(a).

$$f(x) = x\sqrt{2x+1}.$$

(b).

$$f(x) = \pi^2 + 3x - \frac{1}{x}.$$

(c).

$$f(x) = \frac{\sin x}{\cos x + 1}.$$

(d).

$$\tan\left(e^{3x}\right).$$

(e).

$$f(x) = e^{\frac{2}{x}}.$$

(f).

$$f(x) = e^{\sec x}.$$

(g).

$$f(x) = \ln\left(\frac{x^3}{(x+1)^2}\right).$$

(h).

$$f(x) = x\ln(\cos x).$$

***Answer*.**

(a).

$$\begin{aligned}\left(x\sqrt{2x+1}\right)' &= (x)'\cdot\sqrt{2x+1} + x\cdot\left(\sqrt{2x+1}\right)' \\ &= 1\cdot\sqrt{2x+1} + x\cdot\frac{1}{2\sqrt{2x+1}}\cdot(2x+1)' \\ &= \sqrt{2x+1} + x\cdot\frac{1}{2\sqrt{2x+1}}\cdot 2 \\ &= \sqrt{2x+1} + \frac{x}{\sqrt{2x+1}} \\ &= \frac{\sqrt{2x+1}\cdot\sqrt{2x+1}}{\sqrt{2x+1}} + \frac{x}{\sqrt{2x+1}} \\ &= \frac{2x+1+x}{\sqrt{2x+1}} \\ &= \frac{3x+1}{\sqrt{2x+1}}.\end{aligned}$$

(b).

$$\left(\pi^2 + 3x - \frac{1}{x}\right)' = 0 + (3x)' - \left(x^{-1}\right)' = 3 - (-1)\cdot x^{-2} = 3 + x^{-2} = 3 + \frac{1}{x^2}.$$

(c).

$$\begin{aligned}\left(\frac{\sin x}{\cos x + 1}\right)' &= \frac{(\cos x)(\cos x + 1) - (\sin x)(-\sin x)}{(\cos x + 1)^2} \\ &= \frac{\cos^2 x + \cos x + \sin^2 x}{(\cos x + 1)^2}\end{aligned}$$

$$= \frac{\cos x + 1}{(\cos x + 1)^2}$$
$$= \frac{1}{\cos x + 1}.$$

(d).
$$\left(\tan\left(e^{3x}\right)\right)' = \sec^2\left(e^{3x}\right)\cdot e^{3x}\cdot 3 = 3e^{3x}\sec^2\left(e^{3x}\right).$$

(e).
$$\left(e^{\frac{2}{x}}\right)' = e^{\frac{2}{x}}\cdot\left(\frac{2}{x}\right)' = e^{\frac{2}{x}}\cdot\left(2x^{-1}\right)' = e^{\frac{2}{x}}\cdot(-1)\cdot x^{-2} = -\frac{2e^{\frac{2}{x}}}{x^2}.$$

(f).
$$(e^{\sec x})' = e^{\sec x}\cdot(\sec x)' = e^{\sec x}\tan x\sec x.$$

(g).
$$\begin{aligned}\left(\ln\left(\frac{x^3}{(x+1)^2}\right)\right)' &= \frac{1}{\frac{x^3}{(x+1)^2}}\cdot\left(\frac{x^3}{(x+1)^2}\right)' \\ &= \frac{(x+1)^2}{x^3}\cdot\frac{3x^2\cdot(x+1)^2 - x^3\cdot 2(x+1)}{\left((x+1)^2\right)^2} \\ &= \frac{(x+1)^2}{x^3}\cdot\frac{x^2(x+1)(3(x+1)-2x)}{(x+1)^4} \\ &= \frac{(3x+3-2x)}{x(x+1)} \\ &= \frac{x+3}{x(x+1)}.\end{aligned}$$

(h).
$$\begin{aligned}(x\ln(\cos x))' &= (x)'\cdot\ln(\cos x) + x\cdot(\ln(\cos x))' \\ &= \ln(\cos x) + x\cdot\frac{1}{\cos x}\cdot(-\sin x) \\ &= \ln(\cos x) - x\tan x.\end{aligned}$$

Topic 2.C. Implicit differentiation.

Problem 7. *Let*
$$x^2y + y^3 = 2.$$
(a). Use implicit differentiation to find y' at $(-1, 1)$.
(b). Find the equation of the tangent at $(-1, 1)$.

Answer.
(a). Differentiate with respect to x to both sides of the equation:
$$\left(x^2\right)' y + x^2y' + 3y^2\cdot y' = 0, \quad 2xy + \left(x^2 + 3y^2\right) y' = 0.$$

Then
$$\left(x^2 + 3y^2\right) y' = -2xy.$$

Hence,
$$y' = -\frac{2xy}{x^2 + 3y^2}.$$

At $(-1,1)$,

$$y' = -\frac{2\cdot(-1)\cdot 1}{(-1)^2 + 3\cdot 1^2} = \frac{1}{2}.$$

(b). The tangent at $(-1,1)$ has slope $\frac{1}{2}$, and equation

$$y - 1 = \frac{1}{2}\left(x - (-1)\right).$$

That is,

$$y = \frac{1}{2}x + \frac{3}{2}.$$

Topic 3. Applications of differentiation

Topic 3.A. Related rates.

Problem 8.

(a). A spherical balloon is being filled with air so that its volume is increasing at a rate of $8\ \mathrm{in}^3/\mathrm{s}$. *How fast is the radius of the balloon growing, when the radius is* 2 in*? (Hint: the volume of a ball with radius r is $V = \frac{4}{3}\pi r^3$.)*

(b). A dolphin heading north at a rate of 4 mi/hr *passes a boat going east at a rate of* 3 mi/hr*. If it has been* 2 hours *since they passed, how fast is the distance between the boat and the dolphin increasing?*

(c). Sand falls into a pile at a rate of $6\ \mathrm{ft}^3/\mathrm{min}$. *The pile is a cone whose radius is always* 2 *times the height. How fast is the height increasing, when the height is* 3 ft*? (Hint: The volume of a cone with radius of the base r and height h is $V = \frac{1}{3}\pi r^2 h$.)*

***Answer*.**

(a). Differentiate $V = \frac{4}{3}\pi r^3$ with respect to t:

$$V' = 4\pi r^2 \cdot r'.$$

Since $V' = 8$ and $r = 2$,

$$8 = 4\pi \cdot 2^2 \cdot r'.$$

Then

$$r' = \frac{1}{2\pi}\ \mathrm{in/s}.$$

(b). Let x be the distance between the dolphin and the point where they passed. Let y be the distance between the boat and the point where they passed. Then the distance between them, denoted by z, satisfies that

$$z^2 = x^2 + y^2,$$

by the Pythagorean theorem. Differentiate with respect to t:

$$2z\cdot z' = 2x\cdot x' + 2y\cdot y'.$$

Then

$$z' = \frac{x}{z}\cdot x' + \frac{y}{z}\cdot y'.$$

Since $x' = 4$ and $y' = 3$, after 2 hours since they passed,

$$x = 4\cdot 2 = 8, \quad y = 3\cdot 2 = 6, \quad \text{and} \quad z = \sqrt{x^2+y^2} = \sqrt{8^2+6^2} = 10.$$

Then

$$\frac{dz}{dt} = \frac{x}{z}\cdot x' + \frac{y}{z}\cdot y' = \frac{8}{10}\cdot 4 + \frac{6}{10}\cdot 3 = 5\ \mathrm{mi/hr}.$$

(c). Since the radius is always 2 times the height, $r = 2h$,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (2h)^2 h = \frac{4}{3}\pi h^3.$$

Differentiate with respect to t:

$$V' = 4\pi h^2 \cdot h'.$$

Since $h = 3$ and $V' = 6$,

$$6 = 4\pi \cdot 3^2 \cdot h'.$$

Then

$$h' = \frac{1}{6\pi} \text{ ft/min}.$$

Topic 3.B. Mean value theorem.

Theorem (Mean value theorem)**.** *If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 9. *Let $f(x) = \sqrt{x}$ on $[1, 9]$.*
(a). State the mean value theorem for $f(x)$.
(b). Find c in $(1, 9)$ which satisfies the conclusion in Part (a).

Answer.

(a). The function $f(x) = \sqrt{x}$ is continuous on $[1, 9]$ and differentiable on $(1, 9)$. There is at least one point c in $(1, 9)$ such that

$$f'(c) = \frac{f(9) - f(1)}{9 - 1}.$$

(b). First,

$$\frac{\sqrt{9} - \sqrt{1}}{9 - 1} = \frac{1}{4}.$$

Next,

$$f'(x) = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}.$$

Set

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}, \quad 2\sqrt{c} = 4, \quad \sqrt{c} = 2.$$

Then

$$c = 4.$$

Topic 3.C. First-order derivative test and optimization.

Theorem (First-order derivative test)**.**

(i).

$$f \text{ is } \begin{cases} \text{increasing} & \text{if } f' > 0, \\ \text{decreasing} & \text{if } f' < 0. \end{cases}$$

*The points where $f' = 0$ are called the **critical** points.*

(ii). At a critical point, f may attain a maximum or a minimum.

Problem 10. *Find the absolute maximum and absolute minimum of $f(x) = x\sqrt{3 - x}$ on $[-1, 3]$.*

Answer.

- **Step 1**. Find the critical points of the function.

$$\begin{aligned} f'(x) &= \left(x\sqrt{3-x}\right)' \\ &= (x)'\sqrt{3-x} + x\cdot\left(\sqrt{3-x}\right)' \\ &= \sqrt{3-x} + x\cdot\frac{1}{2\sqrt{3-x}}\cdot(3-x)' \\ &= \sqrt{3-x} - \frac{x}{2\sqrt{3-x}} \\ &= \frac{\sqrt{3-x}\cdot 2\sqrt{3-x} - x}{2\sqrt{3-x}} \\ &= \frac{2(3-x)-x}{2\sqrt{3-x}} \\ &= \frac{6-2x-x}{2\sqrt{3-x}} \\ &= \frac{6-3x}{2\sqrt{3-x}} = 0. \end{aligned}$$

  Then

$$6-3x=0.$$

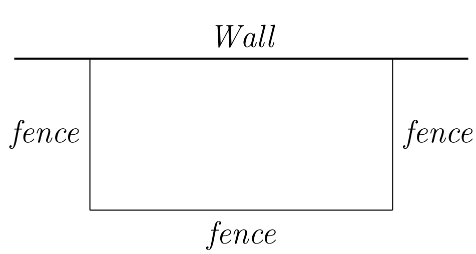
  The critical point is $x=2$.
- **Step 2**. Evaluate the function at the critical point 2 and at the boundary points $-1, 3$.

$$\begin{aligned} f(2) &= 2\cdot\sqrt{3-2} = 2. \\ f(-1) &= -1\cdot\sqrt{3-(-1)} = -2. \\ f(3) &= 3\cdot\sqrt{3-3} = 0. \end{aligned}$$

  The absolute maximum is 2 attained at $x=2$, and the absolute minimum is -2 attained at $x=-1$.

Problem 11.

(a). What two positive numbers with a product of 6 *have the greatest possible sum?*
(b). What rectangle with a perimeter of 12 *has the largest area?*
(c). A rectangular fence is built against a wall, so that there are three sides to the fence. The fence costs \$5 *per* foot *to build. Find the maximum area of the fence, if you have* \$400 *to spend.*



Answer.

(a).
- **Step 1**. Set up the objective function: Let x, y are the two positive numbers. Then the sum $S = x + y$.
  Step 2. Use the condition to eliminate one variable: Since $xy = 6$, $y = \frac{6}{x}$. Then
  $$S = x + \frac{6}{x}.$$
  Step 3. Find the critical point and evaluate the function at that point:
  $$S' = \left(x + \frac{6}{x}\right)' = \left(x + 6x^{-1}\right)' = 1 - 6x^{-2} = 1 - \frac{6}{x^2} = 0.$$
  Then $x^2 = 6$ and $x = \sqrt{6}$ (since it is positive). Hence,
  $$S = \sqrt{6} + \frac{6}{\sqrt{6}} = \sqrt{6} + \sqrt{6} = 2\sqrt{6}.$$
  That is, the largest sum is $2\sqrt{6}$, attained at $x = \sqrt{6}$ and $y = \frac{6}{\sqrt{6}} = \sqrt{6}$.

(b).
- **Step 1**. Set up the objective function: Let w, l be the dimensions of the rectangle. Then the area $A = wl$.
- **Step 2**. Use the condition to eliminate one variable: Since the perimeter $2w + 2l = 12$,
  $$l = \frac{12 - 2w}{2} = 6 - w.$$
  Then
  $$A = wl = w(6 - w) = 6w - w^2.$$
- **Step 3**. Find the critical point and evaluate the function at that point:
  $$A'(w) = 6 - 2w = 0.$$
  Then $w = 3$. Hence,
  $$A(3) = 3 \cdot (6 - 3) = 9.$$
  That is, the maximum area is 9, attained at $w = 3$ and $l = 6 - 3 = 3$.

(c).
- **Step 1**. Set up the objective function: Let x (vertical) and y (horizontal) be the dimensions of the fence. Then the area $A = xy$.
- **Step 2**. Use the condition to eliminate one variable: Since the total length of fence
  $$2x + y = 80 \left(= \frac{400}{5}\right),$$
  $y = 80 - 2x$. Then
  $$A = xy = x(80 - 2x) = 80x - 2x^2.$$
- **Step 3**. Find the critical point and evaluate the function at that point:
  $$A' = \left(80x - 2x^2\right)' = 80 - 4x = 0.$$
  Then $x = 20$. Hence,
  $$A = 20(80 - 2 \cdot 20) = 800 \text{ ft}^2.$$
  That is, the maximum area is 800, attained at $x = 20$ and $y = 80 - 2 \cdot 20 = 40$.

Topic 3.D. Second-order derivative test and graphing.

Theorem (Second-order derivative test)**.**

(i). Second-order derivative test:

$$f \text{ is } \begin{cases} \text{concave up} & \text{if } f'' > 0, \\ \text{concave down} & \text{if } f'' < 0. \end{cases}$$

*The points where $f'' = 0$ are called the **inflection** points.*

(ii). The points for which $f' = 0$ are called the critical points. At each critical point,

$$f \text{ attains } \begin{cases} \text{a local minimum at } x & \text{if } f''(x) > 0, \\ \text{a local maximum at } x & \text{if } f''(x) < 0. \end{cases}$$

Problem 12. *Let*

$$f(x) = 3x^2 - x^3.$$

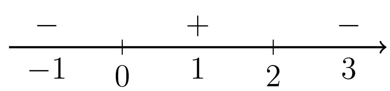
(a). Find the critical points and the intervals on which f is increasing and decreasing.
(b). Find the inflection points and the intervals on which f is concave up and concave down.
(c). List any x values where a local maximum occurs.
(d). List any x values where a local minimum occurs.
(e). Sketch the graph of $f(x)$.

***Answer*.**

(a). Compute that

$$f'(x) = 6x - 3x^2 = 3x(2 - x) = 0.$$

The critical points are $x = 0, 2$.

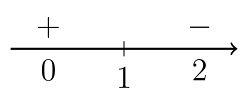


Test at $x = -1$: $f'(-1) = 3 \cdot (-1) \cdot (2 - (-1)) < 0$.
Test at $x = 1$: $f'(1) = 3 \cdot 1 \cdot (2 - 1) > 0$.
Test at $x = 3$: $f'(3) = 3 \cdot 3 \cdot (2 - 3) < 0$.
Hence, f is increasing on $(0, 2)$, and is decreasing on $(-\infty, 0) \cup (2, \infty)$.

(b). Compute that

$$f''(x) = 6 - 6x = 6(1 - x) = 0.$$

The inflection point is $x = 1$.



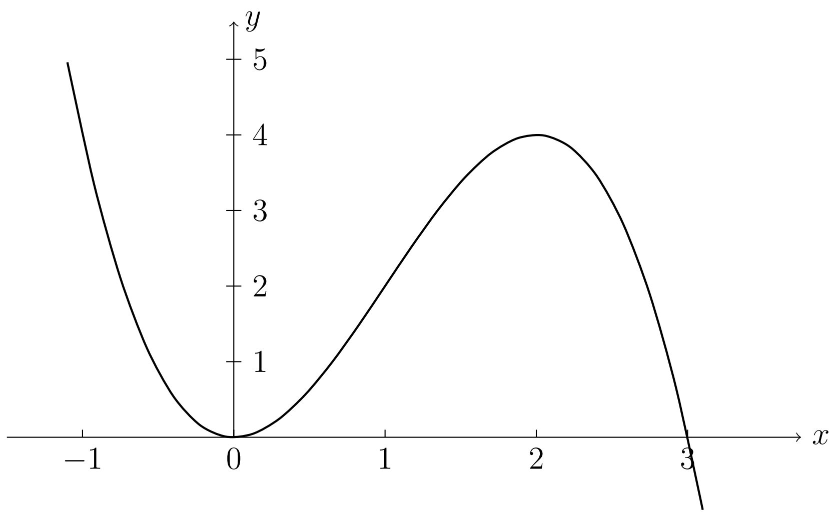
Test at $x = 0$: $f''(0) = 3 \cdot (1 - 0) > 0$.
Test at $x = 2$: $f''(2) = 3 \cdot (1 - 2) < 0$.
Hence, f is concave up on $(-\infty, 1)$, and is concave down on $(1, \infty)$.

(c). At the critical point $x = 2$, $f''(2) = -6 < 0$. Hence, f attains a local maximum $f(2) = 4$ at $x = 2$.

(d). At the critical point $x = 0$, $f''(0) = 6 > 0$. Hence, f attains a local minimum $f(0) = 0$ at $x = 0$.

(e). We sketch the curve from left to right:

- The curve begins from the top-left corner (since $y \to \infty$ as $x \to -\infty$) and is decreasing till $x = 0$.
- The curve is increasing from $x = 0$ to $x = 2$ (with y-intercept $(0,0)$).
- The curve is decreasing from $x = 2$ to the bottom-right corner (since $y \to -\infty$ as $x \to \infty$).



Problem 13. *Let*

$$f(x) = \frac{1}{x^2+1}.$$

(a). Find the critical points and the intervals on which f is increasing and decreasing.
(b). Find the inflection points and the intervals on which f is concave up and concave down.
(c). List any x values where a local extreme value occurs.
(d). Find the horizontal asymptote.
(e). Graph the function.

Answer.
(a). Set

$$f'(x) = \left(\frac{1}{x^2+1}\right)' = \left(\left(x^2+1\right)^{-1}\right)' = (-1)\cdot\left(x^2+1\right)^{-2}\cdot 2x = -2x\left(x^2+1\right)^{-2} = -\frac{2x}{\left(x^2+1\right)^2} = 0.$$

Then $2x = 0$. The critical point is $x = 0$.
Notice that $x^2 + 1 > 0$.
If $f'(x) = -\frac{2x}{(x^2+1)^2} > 0$, then $x < 0$. That is, f is increasing on $(-\infty, 0)$.
If $f'(x) = -\frac{2x}{(x^2+1)^2} < 0$, then $x > 0$. That is, f is decreasing on $(0, \infty)$.

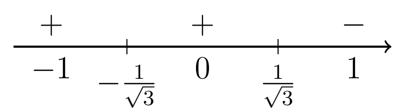
(b). Set

$$\begin{aligned} f''(x) &= \left(-\frac{2x}{\left(x^2+1\right)^2}\right)' \\ &= -\frac{(2x)' \cdot \left(x^2+1\right)^2 - 2x \cdot \left(\left(x^2+1\right)^2\right)'}{\left((x^2+1)^2\right)^2} \\ &= -\frac{2 \cdot \left(x^2+1\right)^2 - 2x \cdot 2\left(x^2+1\right) \cdot 2x}{\left(x^2+1\right)^4} \end{aligned}$$

$$= -\frac{2\left(x^2+1\right)^2 - 8x^2\left(x^2+1\right)}{\left(x^2+1\right)^4}$$
$$= -\frac{2\left(x^2+1\right)\left(\left(x^2+1\right)-4x^2\right)}{\left(x^2+1\right)^4}$$
$$= -\frac{2\left(-3x^2+1\right)}{\left(x^2+1\right)^3}$$
$$= \frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3}$$
$$= \frac{6x^2-2}{\left(x^2+1\right)^3} = 0.$$

Then $6x^2 - 2 = 0$, $x^2 = \frac{2}{6} = \frac{1}{3}$. The inflection points are $x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Since $x^2 + 1 > 0$, the sign of $f''(x) = \frac{6x^2-2}{(x^2+1)^3}$ is the one of $6x^2 - 2$.



Test at $x = -1$: $f''(-1) > 0$.
Test at $x = 0$: $f''(0) < 0$.
Test at $x = 1$: $f''(1) > 0$.
Hence, f is concave up on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$, and is concave down on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(c). At the critical point $x = 0$,

$$f''(0) = \frac{6 \cdot 0^2 - 2}{(0^2+1)^3} = -2 < 0.$$

Hence, f attains a local maximum at $x = 0$.

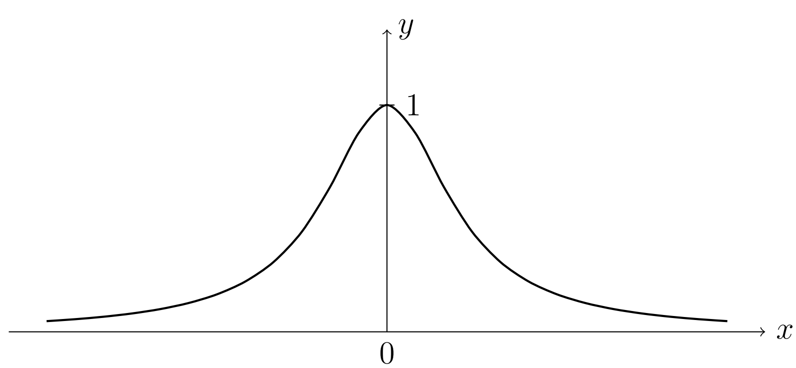
(d). Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2+1} = 0 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2+1} = 0,$$

f has a horizontal asymptote $y = 0$.

(e). We sketch the curve from left to right:

- The curve begins from the bottom-left corner (since $y \to 0$ as $x \to -\infty$) and is increasing till $x = 0$ (with y-intercept $(0, 1)$).
- The curve is decreasing from $x = 0$ to the bottom-right corner (since $y \to 0$ as $x \to \infty$).



Topic 4. Integration

Topic 4.A. Indefinite integrals.

Theorem.

(i). Integrals of powers:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c.$$

(ii). Integrals of trigonometric functions:

$$\int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c, \qquad \int \sec^2 x\,dx = \tan x + c.$$

(iii). Integrals of exponential and logarithmic functions:

$$\int e^x\,dx = e^x + c \quad \textit{and} \quad \int \frac{1}{x}\,dx = \ln|x| + c.$$

Problem 14. *Find the integrals.*

(a).

$$\int \sec^2(5x)\,dx.$$

(b).

$$\int \frac{\cos x}{\sqrt{\sin x + 1}}\,dx.$$

(c).

$$\int x(x+6)^5\,dx.$$

(d).

$$\int \frac{x}{\sqrt{x-11}}\,dx.$$

(e).

$$\int \left(e^{9x} + 8e^{-x}\right)\,dx.$$

(f).

$$\int \frac{e^{6\sqrt{x}}}{\sqrt{x}}\,dx.$$

(g).

$$\int \frac{e^{\frac{x}{9}}}{e^{\frac{x}{9}} + 1}\,dx.$$

(h).

$$\int \frac{\cos(\ln x)}{11x}\,dx.$$

(i).

$$\int \frac{(\ln x)^3}{x}\,dx.$$

(j).

$$\int \frac{3+\sqrt{x}}{x}\,dx.$$

Answer.

(a). Set $u = 5x$. Then

$$du = 5\,dx \quad \text{and} \quad dx = \frac{1}{5}\,du.$$

Hence,

$$\begin{aligned}\int \sec^2(5x)\,dx &= \int \sec^2(u)\cdot\frac{1}{5}\,du \\ &= \frac{1}{5}\int \sec^2(u)\,du \\ &= \frac{1}{5}\tan(u) + c \\ &= \frac{1}{5}\tan(5x) + c.\end{aligned}$$

(b). Set $u = \sin x + 1$. Then

$$du = \cos x\,dx \quad \text{and} \quad dx = \frac{1}{\cos x}\,du.$$

Hence,

$$\begin{aligned}\int \frac{\cos x}{\sqrt{\sin x + 1}}\,dx &= \int \frac{\cos x}{\sqrt{u}}\cdot\frac{du}{\cos x} \\ &= \int \frac{1}{\sqrt{u}}\,du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2u^{\frac{1}{2}} + c \\ &= 2(\sin x + 1)^{\frac{1}{2}} + c.\end{aligned}$$

(c). Set $u = x + 6$. Then $x = u - 6$ and $du = dx$. Hence,

$$\begin{aligned}\int x(x+6)^5\,dx &= \int (u-6)u^5\,du \\ &= \int u^6\,du - 6\int u^5\,du \\ &= \frac{u^7}{7} - 6\cdot\frac{u^6}{6} + c \\ &= \frac{u^7}{7} - u^6 + c \\ &= \frac{1}{7}(x+6)^7 - (x+6)^6 + c.\end{aligned}$$

(d). Set $u = x - 11$. Then $x = u + 11$ and $du = dx$. Hence,

$$\begin{aligned}\int \frac{x}{\sqrt{x-11}}\,dx &= \int \frac{u+11}{\sqrt{u}}\,du \\ &= \int \left(\frac{u}{\sqrt{u}} + \frac{11}{\sqrt{u}}\right)\,du\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{u}{u^{\frac{1}{2}}}\,du + 11\int \frac{1}{u^{\frac{1}{2}}}\,du \\
&= \int u^{\frac{1}{2}}\,du + 11\int u^{-\frac{1}{2}}\,du \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + 11\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= \frac{2}{3}u^{\frac{3}{2}} + 22u^{\frac{1}{2}} + c \\
&= \frac{2}{3}(x-11)^{\frac{3}{2}} + 22(x-11)^{\frac{1}{2}} + c.
\end{aligned}
$$

(e).

$$\int \left(e^{9x} + 8e^{-x}\right)\,dx = \frac{e^{9x}}{9} + 8\cdot\frac{e^{-x}}{-1} + c = \frac{e^{9x}}{9} + 8\cdot\frac{e^{-x}}{-1} + c.$$

(f). Set $u = 6\sqrt{x}$. Then

$$du = \frac{6}{2\sqrt{x}}\,dx = \frac{3}{\sqrt{x}}\,dx \quad\text{and}\quad dx = \frac{\sqrt{x}}{3}\,du.$$

Hence,

$$
\begin{aligned}
\int \frac{e^{6\sqrt{x}}}{\sqrt{x}}\,dx &= \int \frac{e^u}{\sqrt{x}}\cdot\frac{\sqrt{x}}{3}\,du \\
&= \frac{1}{3}\int e^u\,du \\
&= \frac{1}{3}e^u + c \\
&= \frac{1}{3}e^{6\sqrt{x}} + c.
\end{aligned}
$$

(g). Set $u = e^{\frac{x}{9}} + 1$. Then

$$du = \frac{1}{9}e^{\frac{x}{9}}\,dx \quad\text{and}\quad dx = \frac{9}{e^{\frac{x}{9}}}\,du.$$

Hence,

$$
\begin{aligned}
\int \frac{e^{\frac{x}{9}}}{e^{\frac{x}{9}}+1}\,dx &= \int \frac{e^{\frac{x}{9}}}{u}\cdot\frac{9}{e^{\frac{x}{9}}}\,du \\
&= 9\int \frac{1}{u}\,du \\
&= 9\ln|u| + c \\
&= 9\ln\left|e^{\frac{x}{9}}+1\right| + c.
\end{aligned}
$$

(h). Set $u = \ln x$. Then

$$du = \frac{1}{x}\,dx \quad\text{and}\quad dx = x\,du.$$

Hence,

$$
\begin{aligned}
\int \frac{\cos(\ln x)}{11x}\,dx &= \int \frac{\cos u}{11x}\cdot x\,du \\
&= \frac{1}{11}\int \cos u\,du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{11}\sin u + c \\
&= \frac{1}{11}\sin(\ln x) + c.
\end{aligned}
$$

(i). Set $u = \ln x$. Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad dx = x\,du.$$

Hence,

$$
\begin{aligned}
\int \frac{(\ln x)^3}{x}\,dx &= \int \frac{u^3}{x}\cdot x\,du \\
&= \int u^3\,du \\
&= \frac{1}{4}u^4 + c \\
&= \frac{1}{4}(\ln x)^4 + c.
\end{aligned}
$$

(j).

$$
\begin{aligned}
\int \frac{3+\sqrt{x}}{x}\,dx &= \int \left(\frac{3}{x} + \frac{\sqrt{x}}{x}\right)dx \\
&= 3\int \frac{1}{x}\,dx + \int \frac{x^{\frac{1}{2}}}{x}\,dx \\
&= 3\int \frac{1}{x}\,dx + \int x^{-\frac{1}{2}}\,dx \\
&= 3\ln|x| + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= 3\ln|x| + 2x^{\frac{1}{2}} + c.
\end{aligned}
$$

Problem 15. *A particle has acceleration* $a(t) = 3\sin(5t)$ m/s^2*, initial velocity* $v(0) = 1$ m/s*, and initial position* $s(0) = 12$ m*.*
(a). Find the velocity $v(t)$*.*
(b). Find the position $s(t)$*.*

Answer.
(a). Compute that

$$v(t) = \int a(t)\,dt = \int 3\sin(5t)\,dt = 3\int \sin(5t)\,dt = -\frac{3}{5}\cos(5t) + c.$$

Since $v(0) = 1$,

$$-\frac{3}{5}\cos(5\cdot 0) + c = -\frac{3}{5} + c = 1.$$

Then

$$c = \frac{3}{5} + 1 = \frac{8}{5}.$$

Hence, the velocity

$$v(t) = -\frac{3}{5}\cos(5t) + \frac{8}{5} \text{ m/s}.$$

(b). Compute that

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int \left(-\frac{3}{5}\cos(5t) + \frac{8}{5}\right) dt \\ &= -\frac{3}{5}\cdot\frac{1}{5}\sin(5t) + \frac{8}{5}t + c \\ &= -\frac{3}{25}\sin(5t) + \frac{8}{5}t + c. \end{aligned}$$

Since $s(0) = 12$,

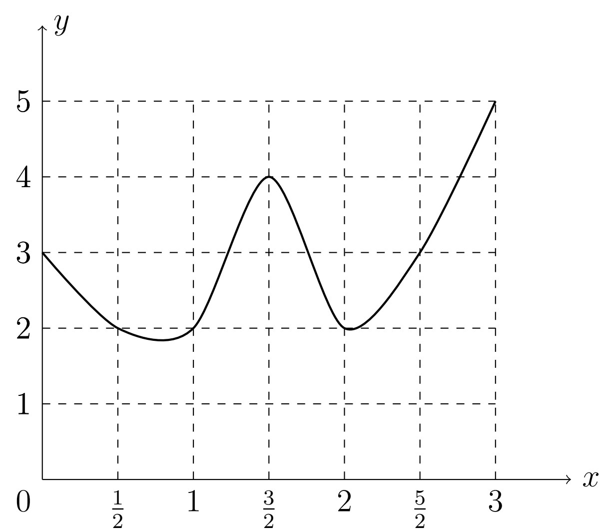
$$-\frac{3}{25}\sin(5\cdot 0) + \frac{8}{5}\cdot 0 + c = c = 12.$$

Hence, the position

$$s(t) = -\frac{3}{25}\sin(5t) + \frac{8}{5}t + 12 \text{ m}.$$

Topic 4.B. Definite integrals.

Problem 16. *Let $y = f(x)$ be defined by the graph. Find the left and right Riemann sums of $y = f(x)$ on $[0, 3]$ with $n = 6$ (the number of sub-intervals).*



Answer. The length of each sub-interval is

$$\frac{3-0}{6} = \frac{1}{2}.$$

The boundary points of the sub-intervals are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.

(a). The left Riemann sum

$$\begin{aligned} & f(0)\cdot\frac{1}{2} + f\left(\frac{1}{2}\right)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f\left(\frac{3}{2}\right)\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} + f\left(\frac{5}{2}\right)\cdot\frac{1}{2} \\ = & \frac{1}{2}(3+2+2+4+2+3) \\ = & 8. \end{aligned}$$

(b). The right Riemann sum

$$\begin{aligned} & f\left(\frac{1}{2}\right)\cdot\frac{1}{2}+f(1)\cdot\frac{1}{2}+f\left(\frac{3}{2}\right)\cdot\frac{1}{2}+f(2)\cdot\frac{1}{2}+f\left(\frac{5}{2}\right)\cdot\frac{1}{2}+f(3)\cdot\frac{1}{2} \\ = & \frac{1}{2}(2+2+4+2+3+5) \\ = & 9. \end{aligned}$$

Problem 17. *Find the integrals using either symmetry or geometry.*

(a).

$$\int_{-2}^{2}(1+|x|)\ dx.$$

(b).

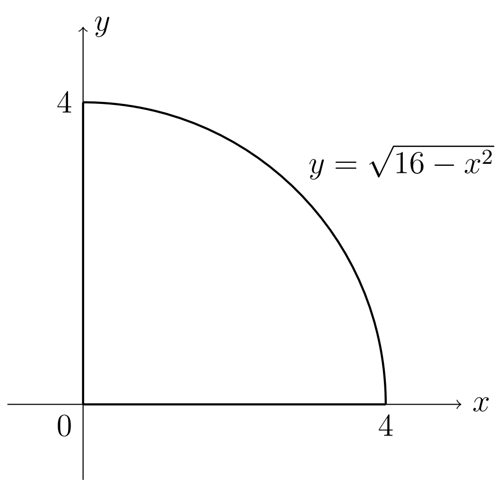
$$\int_{0}^{4}\sqrt{16-x^2}\,dx.$$

Answer.

(a). Using symmetry that $|x|=|-x|$,

$$\begin{aligned} \int_{-2}^{2}(1+|x|)\ dx &= 2\int_0^2(1+x)\,dx \\ &= 2\left[x+\frac{x^2}{2}\right]_0^2 \\ &= 2\left[2+\frac{2^2}{2}\right] \\ &= 8. \end{aligned}$$

(b). Notice that $y=\sqrt{16-x^2}$ on $[0,4]$, that is, $x^2+y^2=16$ with $y\geq 0$, denotes a quarter of the circle centered at the origin and with radius 4.



Hence,

$$\int_0^4\sqrt{16-x^2}\,dx=\frac{1}{4}\cdot\pi\cdot 4^2=4\pi.$$

Topic 4.C. Fundamental theorem of calculus.

Theorem (Fundamental theorem of calculus)**.**

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where F is the antiderivative of f, that is, $F = \int f$.

Problem 18. *Find the integrals using the fundamental theorem of calculus. (Do not integrate.)*

(a).

$$\frac{d}{dx}\int_0^{x^2} \frac{dt}{t^2+5}.$$

(b).

$$\frac{d}{dx}\int_{-x}^{x} \sqrt{5+t^4}\,dt.$$

Answer.

(a). Let

$$f = \frac{1}{t^2+5} \quad \text{and} \quad F = \int f\,dt.$$

Then $F' = f$. By the fundamental theorem of calculus,

$$\int_0^{x^2} \frac{dt}{t^2+5} = F\left(x^2\right) - F(0).$$

Differentiate with respect to x and use the chain rule. Notice that $F(0)$ is a number so the derivative is zero.

$$\frac{d}{dx}\int_0^{x^2} \frac{dt}{t^2+5}\,dt = \left(F\left(x^2\right)\right)' - (F(0))' = f\left(x^2\right)\cdot 2x = \frac{1}{\left(x^2\right)^2+5}\cdot 2x = \frac{2x}{x^4+5}.$$

(b). Let

$$f = \sqrt{5+t^4} \quad \text{and} \quad F = \int f\,dt.$$

Then $F' = f$. By the fundamental theorem of calculus,

$$\int_{-x}^{x} \sqrt{5+t^4}\,dt = F(x) - F(-x).$$

Differentiate with respect to x and use the chain rule.

$$\begin{aligned}
\frac{d}{dx}\int_{-x}^{x} \sqrt{5+t^4}\,dt &= (F(x))' - (F(-x))' \\
&= f(x) - f(-x)\cdot(-1) \\
&= f(x) + f(-x) \\
&= \sqrt{5+x^4} + \sqrt{5+(-x)^4} \\
&= \sqrt{5+x^4} + \sqrt{5+x^4} \\
&= 2\sqrt{5+x^4}.
\end{aligned}$$

Topic 5. Applications of integration

Topic 5.A. Average and work.

Theorem (Average)**.** *The average of f on $[a,b]$ is*

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem 19. *Let*

$$f(x) = \frac{x}{\sqrt{x^2+9}} \quad \text{on } [0,4].$$

(a). Find the average $\overline{f}$.
(b). Find x in $(0,4)$ such that $f(x) = \overline{f}$.

Answer.
(a). Compute that

$$\begin{aligned}
\overline{f} &= \frac{1}{4-0}\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx \quad \left(u = x^2+9,\ du = 2x\,dx,\ dx = \frac{1}{2x}\,du.\right)\\
&= \frac{1}{4}\int_9^{25} \frac{x}{\sqrt{u}}\cdot\frac{1}{2x}\,du \quad (x=0 \Rightarrow u = 9,\ x = 4 \Rightarrow u = 25.)\\
&= \frac{1}{8}\int_9^{25} \frac{1}{\sqrt{u}}\,du\\
&= \frac{1}{8}\int_9^{25} u^{-\frac{1}{2}}\,du\\
&= \frac{1}{8}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_9^{25}\\
&= \frac{1}{4}\left[\sqrt{u}\right]_9^{25}\\
&= \frac{1}{4}\left[\sqrt{25}-\sqrt{9}\right]\\
&= \frac{1}{2}.
\end{aligned}$$

(b). Set

$$\frac{x}{\sqrt{x^2+9}} = \frac{1}{2}, \quad 2x = \sqrt{x^2+9}, \quad 4x^2 = x^2+9, \quad 3x^2 = 9, \quad x^2 = 9.$$

Then

$$x = \pm\sqrt{3}.$$

Hence, $x = \sqrt{3}$ in $(0,4)$ satisfies that $f(x) = \overline{f}$.

Problem 20. *The work required to stretch a spring 1 meter from the equilibrium is 2 joules. Find the work required to stretch the spring from 1 meter to 2 meters?*

Answer.

- **Step 1**. Find the spring constant k. The work required to stretch the spring from $x=0$ to $x=1$ is 2.

$$2 = \int_0^1 kx\,dx = k\left[\frac{x^2}{2}\right]_0^1 = k\cdot\frac{1^2}{2} = \frac{k}{2}.$$

Then

$$k = 4.$$

- **Step 2**. The work required to stretch the spring from $x = 1$ to $x = 2$ is

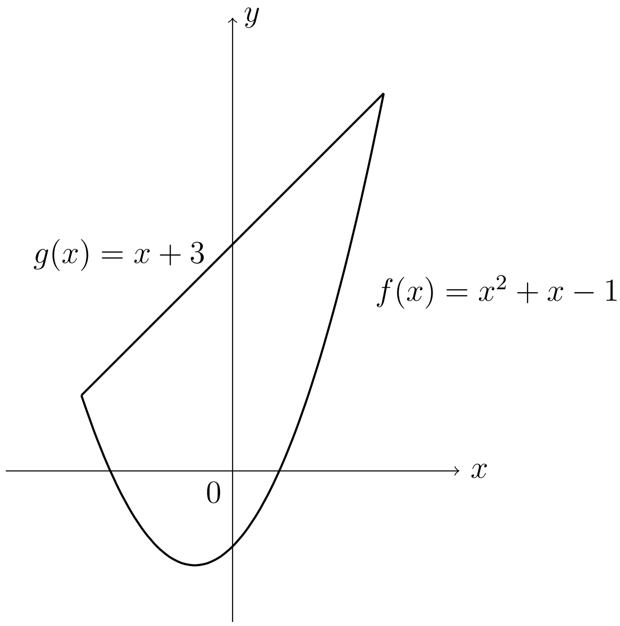
$$\int_1^2 kx\,dx = 4\int_1^2 x\,dx = 4\left[\frac{x^2}{2}\right]_1^2 = 2\left[2^2 - 1^2\right] = 6 \text{ joules.}$$

Topic 5.B. Area between curves.

Theorem. *The area of the region bounded by the upper function $y = f(x)$ and the lower function $y = g(x)$ is*

$$\int_a^b \left(f(x) - g(x)\right)\,dx.$$

Problem 21. *Find the area of the region between $y = x^2 + x - 1$ and $y = x + 3$.*



Answer. Set

$$x^2 + x - 1 = x + 3, \quad x^2 + x - 1 - x - 3 = x^2 - 4 = (x+2)(x-2) = 0.$$

Then $x = -2, 2$. The area is

$$\begin{aligned}
\int_{-2}^2 \left((x+3) - \left(x^2 + x - 1\right)\right)\,dx &= \int_{-2}^2 \left(-x^2 + 4\right)\,dx \\
&= \left[-\frac{x^3}{3} + 4x\right]_{-2}^2 \\
&= \left(\left[-\frac{2^3}{3} + 4\cdot 2\right] - \left[-\frac{(-2)^3}{3} + 4\cdot(-2)\right]\right) \\
&= -\frac{8}{3} + 8 - \frac{8}{3} + 8 \\
&= \frac{32}{3}.
\end{aligned}$$

Topic 5.C. Volume of revolutions.

Theorem (Volume by slices)**.** *Each slice is orthogonal to the rotating axis.*

(i). Let R be bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. The volume of the solid by revolving R around the x-axis is

$$\int_a^b \pi \left(f(x)^2 - g(x)^2\right)\, dx,$$

where $f(x)$ is the outer radius and $g(x)$ is the inner radius of the annulus slice.

(ii). Let R be bounded by $x = f(y)$, $x = g(y)$, $y = a$, and $y = b$. The volume of the solid by revolving around R the y-axis is

$$\int_a^b \pi \left(f(y)^2 - g(y)^2\right)\, dy,$$

where $f(y)$ is the outer radius and $g(y)$ is the inner radius of the annulus slice.

Theorem (Volume by shells)**.** *Each shell is parallel to the rotating axis.*

(i). Let R be bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. The volume of the solid by revolving R around the y-axis is

$$\int_a^b 2\pi x \left(f(x) - g(x)\right)\, dx,$$

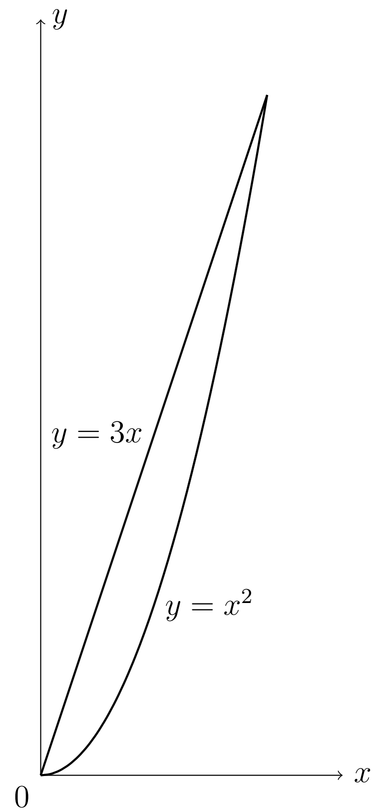
where $2\pi x$ is the circumference and $f(x) - g(x)$ is the height of the cylinder shell.

(ii). Let R be bounded by $x = f(y)$, $x = g(y)$, $y = a$, and $y = b$. The volume of the solid by revolving R around the x-axis is

$$\int_a^b 2\pi y \left(f(y) - g(y)\right)\, dy,$$

where $2\pi y$ is the circumference and $f(y) - g(y)$ is the height of the cylinder shell.

Problem 22. *Let R be the region bounded by $y = 3x$ and $y = x^2$.*



(a). Find the volume given by revolving R around the x-axis.
(b). Find the volume given by revolving R around the y-axis.

Answer. Set

$$3x = x^2, \quad 3x - x^2 = x(3-x) = 0.$$

Then $x = 0, 3$.

(a). **Volume by slices**.

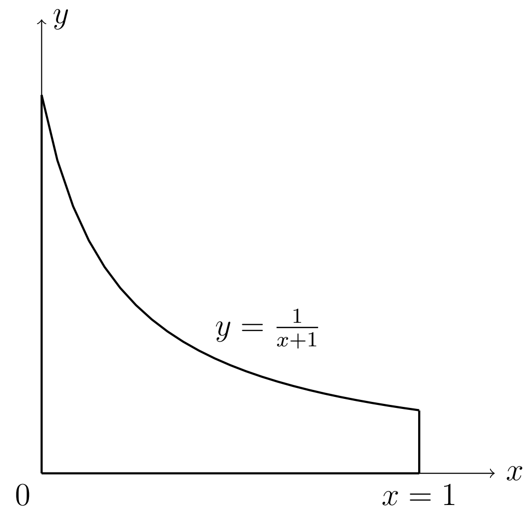
$$\begin{aligned}
\int_0^3 \pi \left((3x)^2 - \left(x^2\right)^2\right) dx &= \pi \int_0^3 \left(9x^2 - x^4\right) dx \\
&= \pi \left[\frac{9x^3}{3} - \frac{x^5}{5}\right]_0^3 \\
&= \pi \left[3 \cdot 3^3 - \frac{3^5}{5}\right] \\
&= \pi \cdot 3^4 \left[1 - \frac{3}{5}\right] \\
&= \pi \cdot 81 \cdot \frac{2}{5} \\
&= \frac{162}{5}\pi.
\end{aligned}$$

(b). **Volume by shells**.

$$\int_0^3 2\pi x \left(3x - x^2\right) dx \;=\; 2\pi \int_0^3 \left(3x^2 - x^3\right) dx$$

$$
\begin{aligned}
&= 2\pi \left[\frac{3x^3}{3} - \frac{x^4}{4}\right]_0^3 \\
&= 2\pi \left[3^3 - \frac{3^4}{4}\right] \\
&= 2\pi \cdot 3^3 \left[1 - \frac{3}{4}\right] \\
&= 2\pi \cdot 27 \cdot \frac{1}{4} \\
&= \frac{27}{2}\pi.
\end{aligned}
$$

Problem 23. *Let R be the region bounded by $y = \frac{1}{x+1}$, $y = 0$, $x = 0$, and $x = 1$.*



(a). Find the volume given by revolving R around the x-axis.
(b). Find the volume given by revolving R around the y-axis.

Answer.

(a). **Volume by slices.**

$$
\begin{aligned}
&\int_0^1 \pi \left(\left(\frac{1}{x+1}\right)^2 - 0^2\right) dx \\
&= \pi \int_0^1 \frac{1}{(x+1)^2}\, dx \quad (u = x+1,\ du = dx.) \\
&= \pi \int_1^2 \frac{1}{u^2}\, du \quad (x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 2.) \\
&= \pi \int_1^2 u^{-2}\, du \\
&= \pi \left[\frac{u^{-1}}{-1}\right]_1^2 \\
&= -\pi \left[\frac{1}{u}\right]_1^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\pi\left[\frac{1}{2}-\frac{1}{1}\right] \\
&= \frac{\pi}{2}.
\end{aligned}
$$

(b). **Volume by shells**.

$$
\begin{aligned}
&\int_0^1 2\pi x\left(\frac{1}{x+1}-0\right)\,dx \\
=\ & 2\pi\int_0^1 \frac{x}{x+1}\,dx \quad (u=x+1,\ du=dx,\ x=u-1.) \\
=\ & 2\pi\int_1^2 \frac{u-1}{u}\,du \quad (x=0\Rightarrow u=1,\ x=1\Rightarrow u=2.) \\
=\ & 2\pi\int_1^2 \left(1-\frac{1}{u}\right)\,du \\
=\ & 2\pi\left[u-\ln|u|\right]_1^2 \\
=\ & 2\pi\left([2-\ln 2]-[1-\ln 1]\right) \\
=\ & 2\pi(1-\ln 2).
\end{aligned}
$$