The Role of Long Division in the K-12 Curriculum

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Abstract
We discuss the role of long division in the K - 12 mathematics curriculum. We begin by reviewing the reasons that most math educators today depreciate the topic and other topics in the curriculum that derive from it, such as polynomial long division or polynomial factorization. Later we show that this view is simply wrong mathematically. The role of long division is not just to divide one rational number by another, but the algorithm itself contains the initial exposure of topics which become crucial in the core applications of mathematics in our society today. Following the introduction, we discuss methods for teaching long division in such a way that the underlying concepts can be understood by students. We then provide more details about the ways in which these concepts develop in later mathematics course, and why they are so important.

Introduction

There is a long standing consensus among those most knowledgeable in mathematics that standard algorithms of arithmetic[1] should be taught to school children. Mathematicians, along with many parents and teachers, recognize the importance of mastering the standard methods of addition, subtraction, multiplication, and division in particular. The profound importance of the long division algorithm and its role as a prerequisite to other parts of K-12 and university level mathematics will be described in this paper.

It is unfortunate that during the past decade, and even before, mathematics education leaders in the United States have called into question the practice of requiring elementary school students to master these standard algorithms. Long division has been especially targeted for de-emphasis, or even elimination from the school curriculum.[2] We hope to make clear why this tendency should be reversed by explaining the importance of the long division algorithm on conceptual grounds.

[1] It is often asserted that there really are no "standard" algorithms for arithmetic. Over the years there have been small variations in the exact procedures taught. For example, from Tom Lehrer's song, The New Math, "Consider the folowing subtraction problem which I'll put up here, 342 – 173. Now, remember how we used to do that: 3 from 2 is 9, carry the one, and if your are under 35 or went ot a private school you say 7 from 3 is 6, but if you are over 35 and went to a public school, you say 8 from 4 is 6, and carry the one so you have 169." The variations are minor. The basic processes are the same and students should understand the underlying principles.

[2] Among the reasons are the complaints from many teachers that it simply takes too much class time to develop. But one of the objectives of this paper is to show that the content concealed in this algorithm is worth a considerable amount of class time, and that there are efficient methods for presenting this content which may not be as time consuming as more standard developments.

The trend was perhaps codified in the 1989 *Curriculum and Evaluation Standards for School Mathematics*, published by the National Council of Teachers of Mathematics. This document, often referred to as the NCTM Standards, specifically recommended that long division receive decreased attention in schools along with "practicing tedious pencil-and-paper computations" (pages 21 and 71). The same document (page 8) recommended to the nation's math teachers that "appropriate calculators should be available to all students at all times."

Support for this direction could be found in the popular press. A special issue of *Newsweek* in 1990 [N] carried an article called *Creating Problems* that derogated pencil-and-paper computation. The opening paragraphs offered a narrow view of arithmetic through a hypothetical race between a calculator and a seventh grade student:

> Let us consider two machines, each capable of dividing 1,128 by 36. The first is a pocket calculator. You punch in the numbers, and in a tenth of a second or so, the answer appears in a digital display, with an accuracy of, for all ordinary purposes, 100 percent.
>
> The second is a seventh grader. You give him or her a pencil and a sheet of paper, write out the problem, and in 15 seconds, more or less, there is a somewhat better-than-even chance of getting back the correct answer.
>
> As between them, the choice is obvious. The calculator wins hands down, leaving only the question of why the junior high schools of America are full of kids toiling over long division, an army of adolescents in an endless trudge, carrying digits from column to column.

Later in the same article, Thomas Romberg, professor of curriculum and instruction at the University of Wisconsin, Madison, is quoted as saying, "There isn't anyone out there anymore who makes his living doing long division."

Newsweek and the *NCTM Standards* were far from alone in promoting calculators over paper-and-pencil computations for young school children. Many articles appeared in journals published by the National Council of Teachers of Mathematics (NCTM) calling into question the teaching of the usual arithmetic procedures and offering calculators in their place for the early grades. A complete survey is beyond the scope of this paper, but a typical view was expressed in a 1994 article in the *Arithmetic Teacher,* when it admonished that, "...the widespread availability of calculators has made traditional skill with paper-and-pencil computational algorithms, and therefore much of the current school mathematics curriculum obsolete..." [AT]. An article published the same year in *Education Week* and written by a mathematics consultant for the Connecticut Department of Education went even further:

> It's time to recognize that, for many students, real mathematical power, on the one hand, and facility with multidigit, pencil-and-paper computational algorithms, on the other, are mutually exclusive. In fact, it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous.[EW]

The view that the four standard arithmetic algorithms are obsolete, superfluous, and perhaps even dangerous was so widespread among education specialists that many elementary school curricula abridged or eliminated them. "Mathland," a K-6 curriculum, is a case in point. Mathland was used by 60% of California's public elementary schools at one point in time, according to its publishers [M]. There are no student textbooks for this NCTMaligned series, but the teacher's manual for each grade urges teachers not to teach the standard arithmetic algorithms, promoting "invented algorithms" by students instead.

By December, 1997 opposition to the long division algorithm was sufficiently well entrenched among education administrators and mathematics education experts in California that the California Academic Standards Commission, an advisory committee to the State Board of Education, recommended a set of K-12 mathematics standards for state-wide adoption which intentionally omitted any requirements for long division except in the case of single digit divisors [OL]. The California State Board of Education rejected this document for a variety of reasons, including this particular shortcoming. In spite of fierce resistance from the education community [Wu], the state board approved a rigorous revision of these standards that was developed with the help of mathematicians at Stanford University.

Recognizing that mathematics education had gone awry, college math teachers and professors responded. More than 100 California mathematics professors endorsed an open letter [OL] in February 1998 supporting the adoption of the far superior current California K-12 mathematics standards. The letter explicitly pointed out that the rejected Commission Standards "fail to require K-12 students ever to master long division when the divisor has more than a single digit." The letter was endorsed by the chairs of the math departments at Stanford University, Caltech, several UC and CSU campuses, the vice president of the American Mathematical Society and a former president of the Mathematical Association of America. Jaime Escalante, portrayed in the movie "Stand and Deliver," also endorsed this letter which explicitly defended long division.

At about the same time, a committee of the American Mathematical Society (AMS), formed for the purpose of representing the views of the AMS to the National Council of Teachers of Mathematics published a report which stressed the mathematical significance of the long division algorithm, as well as addressing other mathematical issues. An excerpt from this report published in the February 1998 issue of the *Notices of the American Mathematical Society* is illuminating [AMS] :

> Standard algorithms may be viewed analogously to spelling: to some degree they constitute a convention, and it is not essential that students operate with them from day one or even in their private thinking; but eventually, as a matter of mutual communication and understanding, it is highly desirable that everyone (that is, nearly everyone--we recognize that there are always exceptional cases) learn a standard way of doing the four basic arithmetic operations. (The standard algorithms need not be absolutely unique, just as there are variant spellings between, say, the U.S. and England, but too much variation leads to difficulties.) We do not think it is wise for students to be left with untested private algorithms for arithmetic operations--such algorithms may only be valid for some subclass of problems. The virtue of standard algorithms--that they are guaranteed to work for all problems of the type they deal with--deserves emphasis.
>
> We would like to emphasize that the standard algorithms of arithmetic are more than just "ways to get the answer"–that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials. The division algorithm is also significant for later understanding of real numbers. For all its virtues, decimal notation suffers a significant drawback over, say, standard

notation for fractions: decimal numbers (meaning decimal fractions with finitely many terms) do not allow division. This can be remedied at the cost of using infinite decimal expansions, but this is a big leap, and the general infinite decimal is not rational. To understand that rational numbers correspond to repeating decimals essentially means understanding the structure of division of decimals as embodied in the division algorithm. We do not see that naive use of calculators can be of much help here: the length of repeat of a decimal will typically be comparable to the size of the denominator, so that 7/23 or 5/29 will not reveal any repeating behavior on standard calculators.

The paragraph above deserves amplification: the long division algorithm is an essential tool for understanding what a real number is. We elaborate on this in the sequel, and explain the important connections of the algorithm to more advanced parts of mathematics.

In spite of the visible support for the standard arithmetic algorithms from professional mathematicians, the trends in the education community against them continued and reached a climax in October of 1999. At that time the U.S. Department of Education, on the advice of the education community, released a list of "exemplary" and "promising" mathematics programs. The standard arithmetic algorithms, and the long division algorithm in particular, did not appear at all, or were drastically abridged, in all of the elementary school curricula on the government's list. In response to these and other serious shortcomings in the Education Department's favored math programs, the authors of this paper together with over 220 other mathematicians, scientists, and education scholars and leaders endorsed an open letter of protest [R] to Education Secretary Richard Riley urging him to withdraw the department's list. Among the co-signers were many of the nation's most distinguished and accomplished scientists and mathematicians. Department heads at 16 universities including Caltech, Stanford, Harvard, and Yale, along with two former presidents of the Mathematical Association of America also added their names in support. Among the endorsers are seven Nobel laureates and winners of the Fields Medal, the highest award in mathematics. Nevertheless, the assault on arithmetic from education leaders continues as of this writing, in spite of this overwhelming support from those with the deepest understanding of mathematics.

In the next section we review the base ten structure of our number system in a form that will illuminate a subsequent description of long division.

Place value and Division

Prior to teaching long division a teacher has to be sure that students understand place value. This is more subtle than one might suppose, so we begin with a presentation that will lead to a way to understand division of whole numbers.

Consider the number 946. One way of describing this number (*which has meaning independent of the way we write it)* is by estimating it through decreasing orders of magnitude. Observe first that $900 = 9\times100$ is the largest multiple of 100 that is less than or equal to 946. So that,

$$9 \times 100 \leq 946 < 10 \times 100.$$

Now we make a similar estimation for the tens column: $900 + 4\times10 \leq 946$, while

$946 < 900 + 5 \times 10$. In other words,

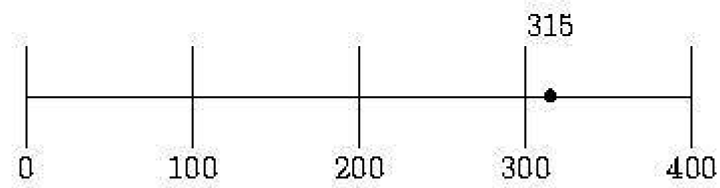
$$4 \times 10 \leq 946 - 900 = 46 < 5 \times 10.$$

Finally, $9 \times 100 + 4 \times 10 + 6 \times 1 \leq 946 < 9 \times 100 + 4 \times 10 + 7 \times 1$, or equivalently.

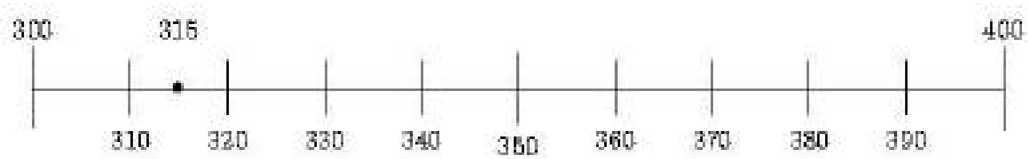
$$6 \leq 946 - 9 \times 100 - 4 \times 10 < 7.$$

Here is the general prescription. An arbitrary whole number with the digit k in the nth decimal means that $k \times 10^n$ is less than or equal to the whole number minus a the sum of multiples of higher powers of ten that are associated to the places to the left of k. But $(k+1) \times 10^n$ is bigger than this number minus the same sum of higher powers of ten associated to the places to the left of k.

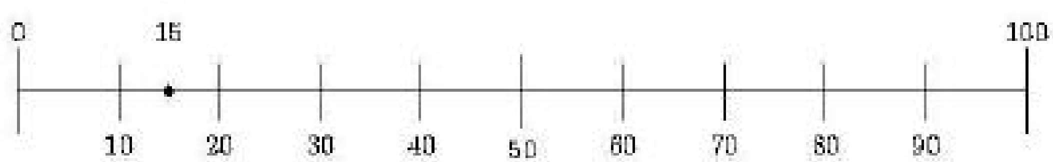
Another way to understand this approach to the base 10 structure of numbers is to use a number line. As an example, the symbol 315 tells us that to identify this number as a point on a number line, 3×100 would be the largest multiple of the largest power of 10 less than or equal to 315 which is either to the left of 315 or is equal to it:



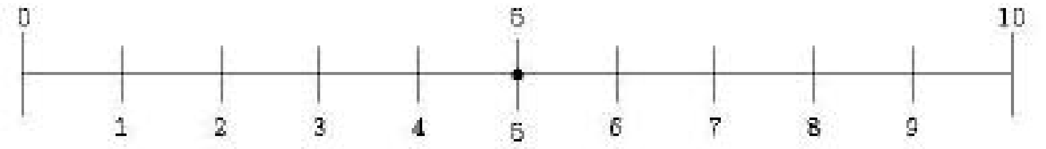
Then 1×10 is the largest multiple of 10 which is either to the left of $315 - 300 = 15$ or equal to it:



The difference $315 - 300$ is located here:



Then the difference which identifies the digit in the ones column, $315 - 3 \times 100 - 1 \times 10 = 5$, is positioned here:



The next step is to extend these ideas to division. Consider the division problem, $946 \div 7$. Following a similar strategy as above, which leads to the standard long division algorithm, our first step is to find the largest multiple of 7×100 that is less than 946. By inspection,

$$1\times (7 \times 100) = 700 \leq 946 < 1400 = 2 \times (7 \times 100)$$

Note that 100 is the highest power of ten that can be used at this step. We next consider multiples of 7×10 and observe that,

$$700 + 3 \times (7 \times 10) = 910 \leq 946 < 980 = 700 + 4 \times (7 \times 10).$$

Analogous to what was done above, this last statement can be expressed as,

$$3 \times (7 \times 10) \leq 946 - 700 < 4 \times (7 \times 10)$$

or, so the calculation would more closely look like in the standard division algorithm,

$$3 \times (7 \times 10) = 210 \leq 246 < 280 = 4 \times (7 \times 10)$$

Finally, we consider multiples of 7×10^0, i.e., multiples of 7 and observe that

$$5 \times 7 \leq 246 - 210 = 36 < 6 \times 7$$

In summary,

$$1\times (7 \times 100) + 3 \times (7 \times 10) + 5 \times 7 \leq 946 < 1\times (7 \times 100) + 3 \times (7 \times 10) + 6 \times 7.$$

The common factor of 7, the divisor, together with the distributive property, allows us to rewrite this as,

$$7 \times (1 \times 100 + 3 \times 10 + 5) < 946 < 7 \times (1 \times 100 + 3 \times 10 + 6)$$

In other words,

$$7 \times 135 \leq 946 < 7 \times 136.$$

Our division problem $946 \div 7$ is solved. The answer is 135 with a remainder of

$$946 - 7 \times 135 = 1.$$

The Long Division Algorithm with Remainder Explained Algebraically

Just as multiplication of counting numbers is based on repeated additions, the inverse operation of division may be understood in terms of repeated subtractions. A useful way to visualize 73 ÷ 6 is to think of 73 objects divided equally among 6 people. When each person receives one object, the number of objects is reduced by 6. If each person receives 5 objects, the initial number 73 is reduced by 5 × 6 = 30. The process ends when the number of objects remaining is less than 6. In the end, each person will receive 12 objects, with one remaining and this can be represented by

$$73 = 6 \times 12 + 1.$$

The division problem 73 ÷ 6 can be posed as the search for whole numbers q and r for which

$$73 = 6 \times q + r \quad \text{where} \quad 0 \leq r < 6.$$

More generally, the division problem a ÷ b is the search for whole numbers q and r for which

$$a = b \times q + r, \quad \text{where} \quad 0 \leq r < b.$$

An efficient way of doing successive subtractions and solving for the numbers q and r is the long division algorithm. For example, in order to solve 5738 ÷ 17, we may think of 17 boys dividing 5738 apples. Rather than asking children to guess "how many hundred apples should each boy take?" the long division algorithm poses the problem in a much more systematic form, though estimating remains a crucial part of the process. Here we initially confront the easier question, "How often does 17 go into 57?"

The fact that

$$3 \times 17 \leq 57 \quad \text{and} \quad 4 \times 17 > 57$$

underlies the first step in this algorithm and leads to

$$\begin{array}{r} 3 \\ 17\overline{)5738} \\ \underline{51} \\ 6 \end{array}$$

Because the digit 3 is in the hundreds column, this first step corresponds to "allocating 300 apples to each boy" or to the fact that

$$5738 - 17 \times 300 = 5738 - 5100 = 638.$$

By way of continuing this process, the next step is based on the fact that

$$3 \times 17 \leq 63 \quad \text{and } 4 \times 17 > 63$$

$$\begin{array}{r@{}l}
 & 33 \\
17 \big) & \overline{5738} \\
 & \underline{51} \\
 & 63 \\
 & \underline{51} \\
 & 12
\end{array}$$

Because this 3 is written in the tens column, it corresponds to "allocating an additional 30 apples to each boy" or to

$$638 - 17 \times 30 = 638 - 510 = 128$$

The final step,

$$\begin{array}{r@{}l}
 & 337 \\
17 \big) & \overline{5738} \\
 & \underline{51} \\
 & 63 \\
 & \underline{51} \\
 & 128 \\
 & \underline{119} \\
 & 9
\end{array}$$

is based on

$$7 \times 17 \leq 128 \quad \text{and } 8 \times 17 > 128$$

and corresponds to "allocating a final 7 apples to each boy" or

$$128 - 17 \times 7 = 128 - 119 = 9.$$

These calculations can be summarized by the equation

$$5738 = 17 \times 337 + 9,$$

indicating that each boy received 337 apples, with 9 left over. It is the requirement that $0 \leq r < 17$ that assures a unique solution to $5736 = 17 \times q + r$.

Converting Fractions to Decimals

The conversion of (finite) decimals to fractions with denominators equal to a power of 10 is straightforward. It involves nothing more than the very definition of a decimal expression. But converting from a fraction to a decimal is more elaborate and involves the division algorithm in an essential way. The justification for this process is more subtle than is often recognized. For example, it is easily computed that

$$\frac{3}{4} = 0.75$$

This is the result of the division calculation

$$\begin{array}{r} .75 \\ 4\overline{)3.00} \\ \underline{2\,8} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

But it is seldom asked, though well worth understanding, why the long division algorithm correctly converts fractions to decimals. Why is it always the case that dividing the numerator of a fraction by its denominator, using the long division algorithm, results in the correct decimal representation of the fraction?

We postpone discussion of the case of fractions with infinite decimal expressions, and focus in this section on fractions reduced to lowest terms whose denominators have no prime factors other than 2 and/or 5. This condition will guarantee that the fraction has a terminating decimal. The reason for this is that the conversion of the fraction a/b to a terminating decimal is possible if and only if there are integers k and n such that

$$\frac{a}{b} = \frac{k}{10^n}$$

To proceed, one needs to find an integer m such that $m \times b = 10^n$ for some whole number n, for convenience the smallest value of n for which this is possible. Then,

$$\frac{m \times a}{m \times b} = \frac{k}{10^n}$$

and $m \times a = k$. Since the Fundamental Theorem of Arithmetic guarantees a unique factorization of any whole number, and

$$10^n = (2 \times 5)^n = 2^n \times 5^n,$$

it follows that the prime factors of b (and m) must be 2 and/or 5.

Returning to the example of 3/4, we observe that it is reduced to lowest terms and that the only prime factor of $4 = 2^2$ is 2. $100 = 10^2$ is the lowest power of 10 with two factors of 2, so to convert 3/4 to a decimal it is necessary to find the whole number k satisfying,

$$\frac{3}{4} = \frac{k}{100}$$

Cross multiplying gives,

$$4k = 3 \times 100$$

or

$$k = \frac{300}{4}$$

This fraction may be reduced to an integer by dividing numerator and denominator by 4,

$$k = \frac{300 \div 4}{4 \div 4}$$

or

$$k = 300 \div 4$$

This tells us that k is the solution to the long division problem $4\overline{)300}$, and accounting for the decimal point, we are led to $4\overline{)3.00}$. The arguments given for this example are general and taken together explain why long division converts fractions of the type considered here to decimals.

Geometry of the Decimal Portion of the Quotient

In this section we use a number line to give an explanation of the long division calculation when the quotient involves decimals. As an example, consider the calculation,

```
   135.14
7)946.00
  7
  24
  21
   36
   35
    10
     7
     30
     28
```

One way to approach the problem "946 divided by 7" geometrically is to find an interval of a certain length L so that if we lay out 7 of them end-to-end, starting at the origin, the right endpoint of the last will be exactly 946, that is to say 946 = 7L. In principle, such an interval exists and can easily be identified. In this case L is the rational number, 946/7, or the length of 946 intervals laid end-to-end, each of length 1/7. However, our goal here is to visualize how the long division algorithm produces L as a decimal expression. It is necessary here and in other cases to acknowledge that L is actually an infinite decimal, and we discuss the significance of this in the next section. But for now, we focus on an iterative process with decreasing length scales.

How might we take the first step to find an interval of this length. To start we could lay out intervals of length 7, one at a time, until adding one more interval would put the last endpoint to the right of 946. We count the number of intervals. Suppose there are k of them. Then seven intervals, each of length k, when placed end to end on the number line, will fall short of 946 by no more than 6 units. The number k is 135 and 7 intervals each of length 135 falls short of 946 by one unit.

It is fruitful to do this portion of the calculation more systematically. First we could use groups of intervals 100 units long (i.e., count by 700's). We can then subtract off the biggest multiple of 700 less than or equal to 946. Then for what is left, count by multiples of 10 intervals at a time (i.e., count by 70's), subtract off, and count by intervals of length seven for what is left. This is how the division algorithm proceeds to the units digit.

This same iterative scheme can be used to explain division past the decimal point. In the case at hand the remainder so far is 1. To continue, lay out intervals of length 0.7 between 0 and 1 analogous to what was done before. For our problem, there is only one such interval. Subtract and we get a remainder of 0.3. Now lay out intervals of length .07 between 0 and 0.3, so there are four. Subtract again and we get a remainder of 0.02.

As this process continues, observe that at each iteration, the remainder must be less than $7/10^n$ for an appropriate integer n, and that n increases by one with each iteration. At the next step we use intervals of length $7/10^{n+1}$ so that the intervals shrink in length by a factor of ten. Each iteration of this process requires an estimate: find the largest integer so that multiplying $7/10^n$ by this integer is less than the remainder at the previous stage.

In an appendix we will give a more detailed description of this process.

Now let us turn to some of the points much further along in mathematics which depend crucially on the skills and insights implicit in the long division process described above.

How Long Division Helps to Explain What a Real Number Is

A fairly standard topic in the middle school curriculum today is the decimal characterization of rational numbers. Rational numbers are eventually repeating, when expanded out as decimals. For example,

$$1/3 = .333...,$$

where the ellipses indicate that the numeral 3 repeats forever. Likewise,

$$1/2 = .500...$$

and

$$611/4950 = .12343434...$$

In the last equation, the digits 34 are repeated without end, and the repeating block in the decimal which represents 1/2 consists only of the digit for zero. In each case there is a repeating block of digits, and this is true for any fraction, i.e., the (infinite) decimal representation for any rational number must eventually have a repeating block.

Clearly the verification of such a general result is beyond the reach of any calculator. This is partly because infinite decimals cannot be displayed on calculator screens, but more importantly calculators are incapable of mathematical reasoning. The proper way to explain this at the middle school level (where questions of convergence are inappropriate), is through the standard long division algorithm. The fraction 1/7 serves as a useful example to illustrate the argument. We perform the division $7\overline{)1.000...}$ in steps.

Following the algorithm to the first stage gives,

$$\begin{array}{l} \quad\ .1 \\ 7\overline{)1.000...} \\ \quad\ \underline{7} \\ \quad\ 3 \end{array}$$

At this point, the algorithm requires that the difference 10 – 7 = 3 is less than 7. If our subtraction had resulted in a number greater than or equal to 7, this would tell us that an error occurred in the calculation, and we would have to start over. At the next stage, we have,

$$\begin{array}{l} \quad\ .14 \\ 7\overline{)1.000...} \\ \quad\ \underline{7} \\ \quad\ \ 30 \\ \quad\ \underline{28} \\ \quad\ \ \ 2 \end{array}$$

Here again, a correct calculation guarantees that the difference 30 – 28 = 2 is less than 7. Carrying the calculation to several more stages gives,

$$\begin{array}{r} .142857 \\ 7\overline{)1.000000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

The last remainder 1 = 50 – 49 tells us something important. The next step in the algorithm is to find $7\overline{)10}$. But this is the same calculation that we started with at the beginning of this division problem, and so the entire sequence in the long division process must repeat with the same differences as before. We therefore conclude that

$$1/7 = .142857\ 142857\ 142857...,$$

where the ellipses indicate that the block of numerals 142857 repeats without end.

From this example, we may easily generalize. Since the remainder at each stage of the standard long division process for $m \div n$ must be between zero and $n - 1$, and since eventually only the zero place digit is brought down to the start of each new stage of the process, it follows that in at most $n - 1$ steps, the process must begin to repeat. If the remainder is ever zero, the result is a terminating decimal, i.e., a repeating decimal whose repeating block consists only of the numeral zero. In the case of $1 \div 7$ carried out above, the process repeats after $6 = 7 - 1$ stages when the remainder 1 reappears. In general, the length of the repeating block for the decimal of the reduced fraction m/n is at most $n - 1$. The length of the repeating block may of course be less than $n - 1$, as is the case for 1/3 or any fraction of the form $m/3$ when m is not divisible by 3.

An exercise perhaps suitable for some students is to deduce that if $1/n$ has a maximal repeating block of length $n - 1$, then for any integer m, m/n is either an integer itself, or has a repeating block in its decimal also of length $n - 1$.

The long division algorithm is the essential tool in establishing that any rational number has a repeating block of digits in its decimal representation. The converse, that any decimal with a repeating block is equal to a rational number, requires a different argument. The idea is most easily demonstrated through examples. Consider the infinite decimal,

$$x = .777...$$

Multiplying both sides of this equation by 10 gives,

$$10x = 7.777...$$

Subtracting eliminates the repeating decimal portion:

$$10x - x = 7.777... - .777...$$

$$9x = 7$$

$$x = 7/9$$

Therefore, $.777... = 7/9$. Here is a more complicated example:

$$x = .576343434...$$

The repeating block in this case consists of 34 (one may also regard the repeating block as 43 with the basic method below unaffected). To express x as a fraction choose multiples of x by two different powers of 10 such that both result in decimals having only the repeating blocks to the right of the decimal point:

$$100{,}000x = 57634.3434...$$

$$1{,}000x = 576.343434...$$

Subtracting gives

$$100{,}000x - 1{,}000x = 57634.3434... - 576.343434...$$

$$99{,}000x = 57634 - 576$$

$$x = 57{,}058/99{,}000$$

Therefore, .576343434... = 57,058/99,000 (which can be further reduced to 28,529/49,500). What we have done is completely general and the procedure can be used to find the sum of any geometric series. This method demonstrates that any infinite repeating decimal represents a rational number and the it even shows how to find the rational number as a fraction. Together with the previous arguments employing the standard long division algorithm, we arrive at an important theorem.

Theorem *Any decimal expression with an infinite repeating block represents a rational number, and any decimal representation of any rational number must have a repeating block (an infinite repeating block consisting of the zero digit is a possibility).*

It is only at this juncture that a sensible explanation (at the middle school level) of irrational numbers is possible. An irrational number is an infinite decimal which does not have a repeating block (or more precisely is a number represented by such a decimal expression). For example,

$$.101001000100001...$$

represents an irrational number. The ellipses in this case indicate that each time "1" appears it is followed by one more "0" than the preceding "1".[3] Clearly this decimal has no repeating block. The set of real numbers consists of all rational and irrational numbers and the definition, at this level, depends crucially on a clear understanding of, and facility with, the standard long division algorithm.

Some widely used middle school mathematics programs not only omit any discussion of the long division algorithm and its role in defining the set of real numbers, but they commit other serious errors as well. It is sometimes asserted that $\sqrt{2}$ is irrational because according to a calculator,

$$\sqrt{2} = 1.4142135624....$$

Unfortunately, this simply says that no repeat is evident up to the precision of a particular calculator. It does not say that no repeat occurs. It usually requires subtle arguments to show that a particular number is irrational. A proof that π is irrational was unknown to mathematicians before the 18th century.

The Role of Long Division in Algebra

A second direct application of long division occurs in studying polynomials.

A common view in math education circles is that not only long division but basic manipulations with polynomials such as factoring, division, and partial fraction decompositions should not appear in the K - 12 mathematics curriculum. We will discuss

[3] Thus the ones appear in exactly the places n(n+1)/2 after the decimal point, for n = 1, 2, 3, …

the negative consequences of making this choice in the next section. For the present, we turn to the process of polynomial long division.

One of the most basic operations with polynomials is long division with remainder. Here the process is almost the same as the process of long division with remainder for natural numbers. The major change is that the role of "highest power of 10 times the divisor" is taken over by "highest power of x times the divisor," and the coefficients are no longer restricted to be integers between 0 and 9. Let us give an example to illustrate the process. We divide $f(x) = x^4 + 2x^3 + 4x + 1$ by $g(x) = x^2 + 1$.

First note that

$$x^2 (x^2 + 1) = x^4 + x^2.$$

This is the only multiple of $x^2 + 1$ by a constant times a power of x so that the product has degree equal to the degree of the original polynomial, f(x), with the same leading coefficient.

Subtracting $x^4 + x^2$ from f(x), gives

$$2x^3 - x^2 + 4x + 1 .$$

In the notation of long division, we have computed,

$$\begin{array}{r|l} & x^2 \\ \hline x^2 + 0x + 1 & x^4 + 2x^3 + 0x^2 + 4x + 1 \\ & \underline{x^4 + 0x^3 + 1x^2} \\ & \quad\quad 2x^3 - 1x^2 + 4x \end{array}$$

Now iterate the process. Multiply $2x (x^2 + 1) = 2x^3 + 2x$, and subtract the product from $2x^3 - x^2 + 4x + 1$ to get

$$-x^2 + 2x + 1.$$

In the notation of long division, we have computed,

$$\begin{array}{r|l} & x^2 + 2x \\ \hline x^2 + 0x + 1 & x^4 + 2x^3 + 0x^2 + 4x + 1 \\ & \underline{x^4 + 0x^3 + 1x^2} \\ & \quad\quad 2x^3 - 1x^2 + 4x \\ & \quad\quad \underline{2x^3 + 0x^2 + 2x} \\ & \quad\quad\quad\quad -1x^2 + 2x + 1 \end{array}$$

As a final step we multiply $(-1)(x^2 + 1) = -x^2 - 1$, and then subtract to get,

$$2x + 2.$$

Because the highest degree term in $2x + 2$, $2x$, has degree less than the degree of $x^2 + 1$, we can no longer repeat the process, so we stop with $2x + 2$ as our remainder.

This entire procedure can be expressed in the standard long division format as,

$$
\begin{array}{r}
x^2 + 2x - 1 \\
x^2 + 0x + 1 \overline{\big) x^4 + 2x^3 + 0x^2 + 4x + 1} \\
\underline{x^4 + 0x^3 + 1x^2} \qquad\qquad \\
2x^3 - 1x^2 + 4x \quad\; \\
\underline{2x^3 + 0x^2 + 2x} \quad\; \\
-1x^2 + 2x + 1 \\
\underline{-1x^2 - 0x - 1} \\
2x + 2
\end{array}
$$

We have shown that,

$$x^4 + 2x^3 + 4x + 1 = (x^2 + 2x - 1)\,(x^2 + 1) + (2x + 2).$$

This procedure is general. Given two polynomials, f(x) and g(x) we can always find a polynomial l(x) and a polynomial r(x) of degree *less than the degree of g(x)* so that

$$f(x) = l(x)g(x) + r(x)$$

using polynomial long division. Note also that this process shows that the degree of l(x) is exactly degree(f(x)) - degree(g(x)) when this number is nonnegative.

A key application of polynomial division is the situation where g(x) is linear: g(x) = x – r. Then the result is

$$f(x) = l(x)(x - r) + c$$

where c is a constant. Now suppose that f(r) = 0. Then substituting x = r on both sides of this equation we get 0 = l(r)(0) + c, or 0 = c, and it follows that, in this case, (x – r) exactly divides f(x) with quotient l(x), which is the same thing as saying that

$$f(x) = l(x)(x - r).$$

Conversely, if f(x) = l(x)(x – r) for some polynomial l(x), it follows immediately that f(r) = 0. Long division for polynomials leads to an important theorem in algebra.

Theorem. *The number r is a root of the polynomial f(x) if and only if f(x) can be factored as a product of x – r and another polynomial.*

This theorem is valid even for complex numbers, and while the theorem can be proved without direct reference to the long division algorithm, it is really the ideas imbedded in that algorithm which lead to this result.

Foundation for More Advanced Applications

There are at least two separate areas where the insights gained from understanding the long division algorithm are crucial, first calculus, and second, applications of polynomials in advanced areas of mathematics and related fields.

Let us discuss calculus first. The student's experience of carrying long division past the decimal point is his or her first experience with infinite processes of any kind converging. At each stage the accuracy of the answer increases approximately by a factor of 10. Compare this with the only other infinite process we can reasonably assume students have experienced, that of counting numbers. At each stage we're no closer to the end than the stage before. Polynomial long division leads in a natural way both to the geometric series and to the priliminary expansion $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + x^3 + \ldots + x^n)$.

The development of polynomial long division up to this point is modeled on the assumption that higher powers of x are larger than lower powers, just as 10^{15} is larger than 10^5. However, the reverse assumption is often equally valid – that higher powers of x become much smaller than lower powers, as, for example happens when x = 0.1. This leads to a second approach to long division for polynomials. For example, consider the problem of dividing 1 by (1 – x). By iterating the long division process, using higher powers of x as remainders, we obtain the expression

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots + x^n \text{ with remainder } x^{n+1.}$$

This gives a basic formula with innumerable applications. For example, it provides the underpinnings of compound interest, and further in the future, functions such as the exponential function. (The exponential function arises by letting the time between compounding periods approach zero, or, as one might say, considering continuous compounding.)

More advanced topics also depend on basic polynomial manipulations of the type described above. An early application in the university curriculum occurs in calculus courses, where students learn how to use the partial fraction decomposition to integrate rational functions (quotients of polynomials). Previous to this the students have learned the rules for integrating simple polynomials and some trigonometric functions, all of which are quite direct, following basically from the definitions and the fundamental theorem of calculus.

But it turns out – unfortunately only much later – that the partial fraction decompositions play a unique and important role in modern applications of mathematics. They form the basis for the techniques needed for using the Laplace Transform in handling systems of linear differential equations, and consequently for many areas of engineering. Indeed, engineers in any area which involves control systems – aeronautical engineering, mechanical engineering, and much of electrical engineering – will confirm that most of what they do depends on Laplace Transforms.

There is reason to believe that students who wish to enter technical areas but have had only minimal experience with polynomials and none with partial fraction techniques are at a severe disadvantage, and at least some are forced to give up their aspirations.

Of course, the applications to the Laplace Transform discussed above are not the only, nor the best known uses of basic polynomial operations in vital applications of mathematics. The other major area of application is linear algebra. Here, manipulations involving the characteristic polynomial are critical in determining the eigenvalues and eigenvectors of linear transformations, which in turn have basic applications to economics, the social sciences, as well as the physical sciences and engineering.

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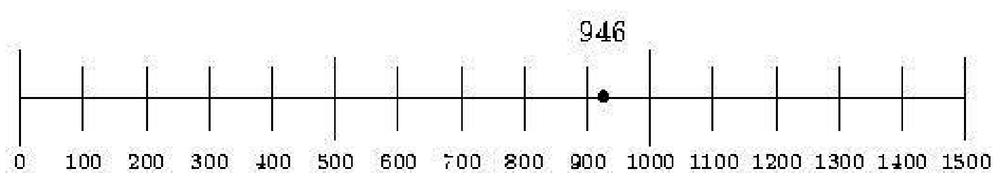
Appendix: A Detailed Geometric Approach to Long-division

Let us consider the problem 946 ÷ 7. Note that the quotient is less than 1000 since $946 < 7\times1000$, so $946 \div 7 < 1000$. Also, the quotient is greater than 100 since $7\times100 < 946$, so $100 = 7\times100 \div 7 < 946 \div 7$.

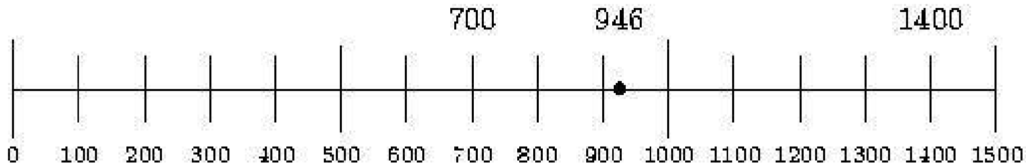
We see that if we want to know which is the smallest power of 10, 10^k, that is greater than $946 \div 7$ we have to find the first time that 7×10^k is greater than 946, *not simply the first time that 10^k is greater than 946.* Similarly, if we want to find the smallest power of 10, 10^s that is bigger than $13{,}123 \div 17$ we have to look for first time that 17×10^s is bigger than 13,123.

We can look at division on the number line as well.

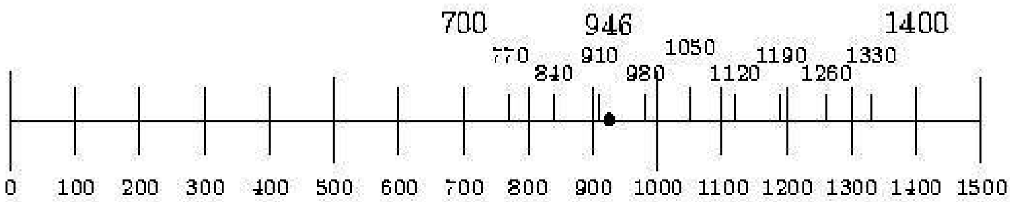
Let us consider the problem of 946 ÷ 7 again. This time we start by placing 946 on the number line.



Next repeat this same picture, but also put in multiples of 700.



Note that 946 is between 700 and 1400. Hence 946 ÷ 7 is between 700 ÷ 7 = 100 and 1400 ÷ 7 = 200. More exactly, 946 ÷ 7 is bigger than 100 but less than 200. Next we take a finer mesh between 700 and 1400. If we divide the distance between 700 and 1400 (which is 700) into 10 equal segments, then each segment will be 70, since 10×70 = 700.



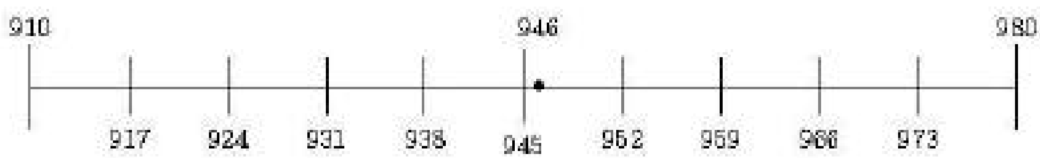
We see that 946 lies between the third and the fourth tic, that is, it lies between 910 and 980. So we have that 7×100 + 3×7×10 is less than 946 while 7×100 + 4×7×10 is greater than 946.

Rewriting, we have that

$$7\times (100 + 3 \times 10) < 946 < 7\times(100 + 4\times 10)$$

So $130 = (100 + 3\times 10) < 946 \div 7 < 140 = (100 + 4\times 10)$.

We can enlarge the picture and repeat the process, concentrating on the interval between 910 and 980, which has length 70. If we divide it into 10 equal segments, each segment will have length 7.



Repeating the argument we see that 946 lies between 945 which is

$$7\times 100 + 7\times 3\times 10 + 7\times 5\times 1$$

or

$$7\times(100 + 3\times 10 + 5\times 1)$$

and 952 which is 7×(100 + 3×10 + 6×1). Consequently, 946 ÷ 7 lies between

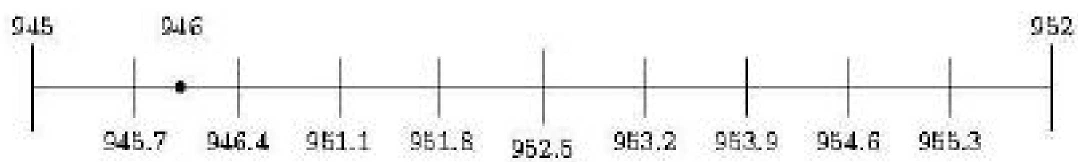
$$135 = (100 + 3\times10 + 5\times1)$$

and

$$136 = (100 + 3\times10 + 6\times1).$$

At this point we could stop and say the difference between 135×7 and 946 is 1, so 946 = 7×135 + 1, and since 0 < 1 < 7 we stop and proclaim that 7 divides 946 a total of 135 times with a remainder of 1. This is appropriate for the first level of development of long-division. One can proceed naturally from this point to the development of the long division algorithm with remainder. It is also possible to continue the discussion of the approximation method for understanding what division is about.

Suppose that we repeat the process. If we break the interval between 945 and 952 which has length 7 into 10 equal parts, each part will have length 7 / 10 or 0.7, and we can expand the picture above as:



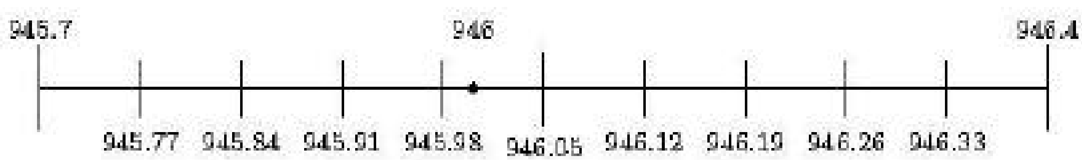
Consequently we have that

$$(7\times135 + 1\times7\times.1) = 945.7 < 946 < (7\times135 + 2\times7\times.1) = 946.4$$

and dividing by 7,

$$135.1 < 946 \div 7 < 135.2.$$

For emphasis one can expand the diagram one more time, noting that the interval between 945.7 and 946.4 has length .7, so that if we break it up into 10 equal segments, each will have length .07:



Consequently, we have that

$$(945.7 + 4\times7\times.01) = 945.98 < 946 < (945.7 + 5\times7\times.01) = 946.05$$

or dividing by 7,

$$(135.1 + 4\times.01) = 135.14 < 946 \div 7 < (135.1 + 5\times.01) = 135.15.$$

At this point we have probably gone far enough. We should not lose sight of the remainder of course, so we should mention that, in fact, since

$$7\times(135.14) = 945.98$$

we can write, more exactly

$$946 = 7\times(135.14) + .02$$

or, to two decimal places $946 \div 7 = 135.14$ with a remainder of .02.

Note that the remainder is always positive, since we *choose to approximate from below* in all cases.

How might one proceed from the analysis of the division process above to a long-division algorithm?

One way would be to say that we don't really need the number line to help us. How did we begin? We decided that we would use gaps of 700's to divide the line up into multiples by 7 of 100, and see which of these was closest to 946 but still less than it. Why did we do that? Because this is the biggest number of the form a power of 10 times 7 which is still less than 946. If we had started by using gaps of 7000, they would have been too big, and if we'd used gaps of 70 they'd have been too small since 10 gaps of 70 only give 700.

As an example, if we'd decided that we wanted to divide 9467 by 7, then we would start with gaps of 7000, and $1\times7000 < 9467 < 2\times7000$ so

$$1000 < 9467 \div 7 < 2000.$$

In any case, we decide that gaps of 700 are what we use to start, and 946 lies in the second gap, *so 946 – 700 must be less than 700*.

NOTE: *this phrase is the critical step. It must be understood!*

At this point, we no longer need to worry about 946, but we can focus on 946 - 700 = 246. What we did, at the next stage, dividing the gap between 700 and 1400 into 10 steps is just the same as dividing the gap between 0 and 700 into 10 steps, finding out where 246 lies in this subdivision (between 3×70 and 4×70), and writing 946 - 700 - 210 = 36.

Now we need not worry about 246 or 946 any more, we only need to worry about 36. So we divide the interval between 0 and 70 into 10 equal steps of length 7, and see that 36 is between 5×7 and 6×7. We then have 946 – 700 – 210 – 35 = 1, or

$$946 = 7\times(100 + 3\times10 + 5\times1) + 1.$$

The analysis of the concept of the algorithm is now complete. What remains is to find a way of implementing it, and at this point we should be able to just write down the long-division algorithm.