

# Geometric drawings of $K_n$ with few crossings

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## Abstract

We give a new upper bound for the rectilinear crossing number  $\overline{cr}(n)$  of the complete geometric graph  $K_n$ . We prove that  $\overline{cr}(n) \leq 0.380559 \binom{n}{4} + \Theta(n^3)$  by means of a new construction based on an iterative duplication strategy starting with a set having a certain structure of halving lines.

**Keywords:** Rectilinear crossing number, crossing number, geometric graph, complete graph, convex quadrilateral.

**AMS Subject Classification:** 05C62, 52A10, 52A22, 52C10.

## 1 Introduction

The *crossing number*  $cr(G)$  of a simple graph  $G$  is the minimum number of edge crossings in any drawing of  $G$  in the plane, where each edge is a simple curve. The *rectilinear crossing number*  $\overline{cr}(G)$  is the minimum number of edge crossings in any *geometric drawing* of  $G$ , that is a drawing of  $G$  in the plane where the vertices are points in general position and the edges are straight segments. The crossing numbers have many applications to Discrete Geometry and Computer Science, see for example [9] and [10].

In this paper we contribute to the problem of determining  $\overline{cr}(K_n)$ , where  $K_n$  denotes the complete graph on  $n$  vertices. Specifically, we construct geometric drawings of  $K_n$  with a small number of crossings. For simplicity we write  $\overline{cr}(n) = \overline{cr}(K_n)$ . We note that a geometric drawing of  $K_n$  is determined by the location of its vertices and two edges cross each other if and only if the quadrilateral determined by their vertices is convex. Thus our drawings also provide constructions of  $n$ -element point sets with small number of convex quadrilaterals determined by the  $n$  points. The problem of finding the asymptotic behavior of  $\overline{cr}(n)$  is also important because of its close relation to Sylvester's four point problem [13]: determine the probability that four points selected uniformly at random on a given domain, form a convex quadrilateral. It is easy to see that  $\{\overline{cr}(n) / \binom{n}{4}\}$  is a non-decreasing sequence bounded above by 1. Thus  $v = \lim_{n \rightarrow \infty} \overline{cr}(n) / \binom{n}{4}$  exists, and Scheinerman and Wilf [12] proved that  $v$  is the infimum, over all open sets  $R$  with finite area, of the probability that four randomly chosen points in  $R$  are in convex position.

When this problem was first investigated, the best lower bounds were obtained by using an averaging argument over subsets whose crossing numbers were known. For example, it is well known that  $\overline{cr}(5) = 1$ , so by counting the crossings generated by every subset of size 5 it is easy to get  $\overline{cr}(n) \geq (1/5) \binom{n}{4}$ . Wagner [14] was the first to use a different approach, he proved  $v \geq 0.3288$ . Then the authors [1] and independently Lovasz et al. [8], used allowable sequences to prove  $v \geq 3/8 = 0.375$ . Lovasz et al. [8] managed to even improve this  $v$  by  $10^{-5}$ , and Balogh and Salazar [5] refined this technique even further to obtain the currently best bound of  $v \geq 0.37533$ .

The history of the improvements on the upper bounds is as follows: In the early seventies Jensen [7] and Singer [11] obtained  $v \leq 7/18 < 0.3888$ , and  $v \leq 5/13 < 0.38462$  respectively. Much later Brodsky et al. [6] constructed point sets with nested non-concentric triangles yielding

$v \leq 6467/16848 < 0.383844$ . Then Aichholzer et al. [3] devised a lense-replacement construction depending on a suitable initial set. They obtained  $v \leq 0.380739$  and later on Jensen provided a better initial set [4], which gave the previously known best bound of  $v \leq 0.38062$ .

In this paper we prove the following,

**Theorem 1**  $\overline{cr}(n) \leq \frac{29969}{78750} \binom{n}{4} + \Theta(n^3) < (0.380559) \binom{n}{4} + \Theta(n^3)$ .

This theorem is based on the following stronger result which may improve the upper bound in the future. To accomplish this we would only need a point-set  $P$  having a halving-line matching (defined next) and with a better  $\binom{n}{4}$  coefficient.

**Theorem 2** *If  $P$  is a  $N$ -element point set in general position, with  $N$  even, and  $P$  has a halving-line matching; then*

$$\overline{cr}(n) \leq \left( \frac{24 \text{cr}(P) + 3N^3 - 7N^2 + (30/7)N}{N^4} \right) \binom{n}{4} + \Theta(n^3).$$

Let  $P = \{p_1, p_2, \dots, p_N\}$  be a general position point set in the plane. We define a *halving line* of  $P$  as a line passing through two points in  $P$  and leaving the same number of points of  $P$  on either side of the line. According to this,  $N$  needs to be even for  $P$  to have a halving line.

Consider a bipartite graph  $G = (P, H)$  where  $H$  is the set of halving lines of  $P$  and  $p \in P$  is adjacent to  $l \in H$  if  $p$  is on the line  $l$ . If there is a matching for  $P$  in the graph  $G$  we say that this matching is a *halving-line matching* of  $P$ . Note that such a matching induces a function on  $P$ ,  $p_i \mapsto p_{f(i)}$ , such that  $p_i p_{f(i)}$  is a halving line of  $P$ .

In general, most even sets  $P$  minimizing the crossing number have a large number of halving lines and consequently most of these sets have a halving-line matching. The only exception seems to be  $P_4$ , the set consisting of a triangle with a point inside. This set achieves  $\overline{cr}(4) = 0$  but it only has three halving lines.

The previously best upper bound, obtained by Aichholzer et al. [3], used a lens-replacement construction yielding

$$\overline{cr}(n) \leq \left( \frac{24 \text{cr}(P) + 3N^3 - 7N^2 + 6N}{N^4} \right) \binom{n}{4} + \Theta(n^3) \quad (1)$$

for any initial point set  $P$  with  $N$  elements,  $N$  even. Jensen (reported in [4]) discovered a set of 36 points with 21175 crossings which happened to be the best initial set known at the time. The bound obtained is  $\overline{cr}(n) \leq (0.38062) \binom{n}{4} + \Theta(n^3)$ . Note that our construction from Theorem 2 always gives a better bound compared to (1) as long as the starting set  $P$  has a halving-line matching.

## 2 The Construction

The construction is based on the following lemma,

**Lemma 3** *If  $P$  is a  $N$ -element point set,  $N$  even, and  $P$  has a halving-line matching; then there is a point set  $Q = Q(P)$  in general position,  $|Q| = 2N$ ,  $Q$  also has a halving-line matching, and  $\overline{cr}(Q) = 16\overline{cr}(P) + (N/2)(2N^2 - 7N + 5)$ .*

**Proof.**  $Q$  is constructed as follows. Each point  $p_i \in P$  will be replaced by a pair of points  $q_{i1}$  and  $q_{i2}$ . Using the function  $f$  induced by the halving-line matching we define

$$q_{i1} = p_i + \varepsilon \frac{p_{f(i)} - p_i}{\|p_{f(i)} - p_i\|} \quad \text{and} \quad q_{i2} = p_i - \varepsilon \frac{p_{f(i)} - p_i}{\|p_{f(i)} - p_i\|},$$

where  $\varepsilon$  is small enough so that all points  $q_{j1}, q_{j2}$  coming from a point  $p_j$  located to the left (right) of  $\overrightarrow{p_i p_{f(i)}}$ ; are also located to the left (right) of the lines  $q_{i1} q_{f(i)1}$  and  $q_{i1} q_{f(i)2}$ . In other words, the

cone with vertex  $q_{i1}$  spanned by the segment  $q_{f(i)1}q_{f(i)2}$  does not contain any of the points  $q_{jx}$  with  $j \neq i, f(i)$ . One way to find such an  $\varepsilon$  is the following: suppose that  $P$  is contained in a disk with diameter  $D$ , since  $P$  is in general position there is  $\delta > 0$  so that the strips of width  $\delta$  centered at the lines  $p_i p_{f(i)}$  have only the points  $p_i$  and  $p_{f(i)}$  with  $P$  in common. The portion contained in the disk of any cone with center in the line  $p_i p_{f(i)}$ , axis  $p_i p_{f(i)}$ , and with an opening angle of  $2 \arctan(\delta/2D)$ ; is a subset of the strip of width  $\delta$  around  $p_i p_{f(i)}$ . Thus we can start with a small arbitrary value  $\varepsilon_0 < \delta$  such that all points  $q_{j1}, q_{j2}$  are inside the disk. Obtain  $\varepsilon_i$  as the distance from  $p_{f(i)}$  to the lines subtending the cone with center  $q_{i1}$ , axis  $p_i p_{f(i)}$ , and angle  $2 \arctan(\delta/2D)$ . And finally set  $\varepsilon$  as the minimum of all the  $\varepsilon_i$ .

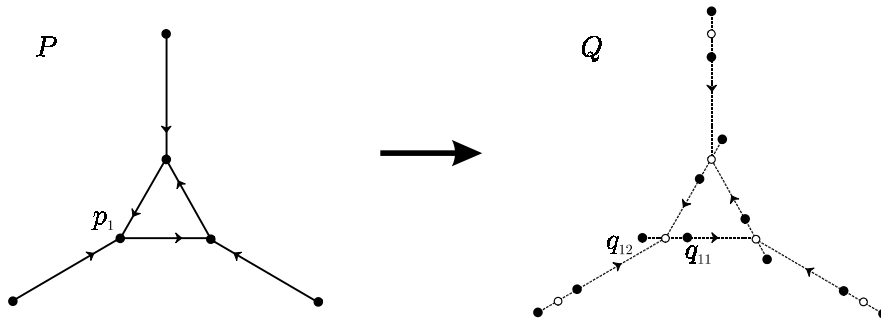


Figure 1: Construction of the set  $Q$

From the halving-line matching definition we deduce that no two points in  $P$  are associated to the same halving line. This, together with the choice of  $\varepsilon$ , guarantees that  $Q$  is in general position whenever  $P$  is in general position. By construction the line  $q_{i1}q_{i2}$  is a halving line of  $Q$ . In addition, since  $q_{i1}$  is in the interior of the triangle  $q_{i2}, q_{f(i)1}, q_{f(i)2}$  the line  $q_{i1}q_{f(i)1}$  (and also the line  $q_{i1}q_{f(i)2}$ ) is a halving line of  $Q$ . Then

$$\{(q_{i1}, \overleftarrow{q_{i1}q_{f(i)1}}) : i = 1, 2, \dots, N\} \cup \{(q_{i2}, \overleftarrow{q_{i1}q_{i2}}) : i = 1, 2, \dots, N\}$$

is a halving-line matching of  $Q$ .

Now we proceed to calculate  $\overline{cr}(Q)$  by counting crossings according to three different types:

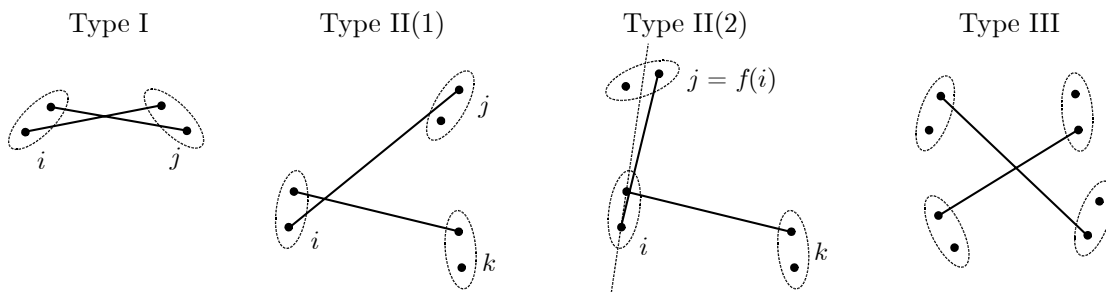


Figure 2: Different types of crossings

**Type I.** Two points in pair  $i$  and two in pair  $j \neq i$ . There are  $\binom{N}{2}$  ways of choosing pairs  $i$  and  $j$  and all of them determine a crossing except when  $j = f(i)$  or  $i = f(j)$ . Since there are exactly  $N$  pairs  $(i, f(i))$ , the total number of crossings in this case is  $\binom{N}{2} - N$ .

**Type II** Two points in pair  $i$  and the two other in pairs  $j$  and  $k$  all pairs distinct. First there are  $N$  choices for the pair  $i$ . Then we have two cases, when  $j, k \neq f(i)$ , and when either  $j$  or  $k$

equals  $f(i)$ . In the first case (see Figure 2, TypeII(1)), to have a crossing, both  $p_j$  and  $p_k$  must be on the same side of the line  $q_{i1}q_{i2}$ . Thus there are  $2\binom{N/2-1}{2}$  ways of choosing  $j$  and  $k$  and then 4 choices for the second indices  $x$  and  $y$  for the points  $q_{jx}$  and  $q_{ky}$ . In the second case (see Figure 2, TypeII(2)) we can assume  $j = f(i)$ . Then there are 2 choices for the second index  $x$  of  $q_{jx}$ . Again, to have a crossing, we need  $q_{jx}$  and  $p_k$  on the same side of  $q_{i1}q_{i2}$ . So there are  $N/2 - 1$  ways of choosing  $k$  and 2 choices for the second index  $y$  of  $q_{ky}$ . The total number of Type II crossings is  $N \left( 8\binom{N/2-1}{2} + 4(N/2 - 1) \right)$ .

**Type III** Each point in a different pair. To have a crossing each of the four pairs must come from a crossing in  $P$ , so there are  $\overline{cr}(P)$  possible pairs, and there are 2 choices for the second index in each pair. Thus there are  $16\overline{cr}(P)$  number of crossings of Type III.

The list of crossings types is complete since by construction none of the segments  $q_{i1}q_{i2}$  participate in any crossing. Adding Types I-III yields the result.  $\blacksquare$

$n$	$\overline{cr}(P_n)$	$\overline{cr}(Q(P_n))$	old bound for $\overline{cr}(2n)$	new bound for $\overline{cr}(2n)$
30	9726	179541	$\leq 179554$	$\leq 179541$
36	21175	381010	$\leq 381020$	$\leq 381010$
48	71028	1239096	$\leq 1239139$	$\leq 1239096$

Table 1: Improved bounds for 60,72,and 96 points.

The strength of this Lemma can be easily seen when it is applied to the set  $P_6$  that minimizes  $\overline{cr}(6) = 3$ . We obtain  $\overline{cr}(12) \leq \overline{cr}(Q) = 153$  which happens to be the correct value of  $\overline{cr}(12)$  (a drawing is shown in Figure 1). It took some effort [3] to find this point set in the past. In addition, if we use the sets  $P_{30}, P_{36}, P_{48}$  obtained by Aicholzer et al., and reported in web page [2], we obtain sets  $P_{60}, P_{72}, P_{96}$  with less number of crossings than those previously known (see Table 1).

$i^{th}$ point = $p_i = (x\text{-coordinate}, y\text{-coordinate})$			
$p_1 = (9259, 16598)$	$p_9 = (28477, 16613)$	$p_{17} = (5141, 23755)$	$p_{25} = (9075, 15320)$
$p_2 = (9763, 16199)$	$p_{10} = (15909, 16415)$	$p_{18} = (9154, 17055)$	$p_{26} = (7921, 13407)$
$p_3 = (9977, 16397)$	$p_{11} = (9446, 15905)$	$p_{19} = (0, 32394)$	$p_{27} = (5206, 8451)$
$p_4 = (10248, 16225)$	$p_{12} = (9540, 16541)$	$p_{20} = (6820, 20921)$	$p_{28} = (9121, 15603)$
$p_5 = (10666, 16385)$	$p_{13} = (9262, 16627)$	$p_{21} = (9949, 16415)$	$p_{29} = (480, 0)$
$p_6 = (12849, 16335)$	$p_{14} = (9282, 16947)$	$p_{22} = (9355, 16177)$	$p_{30} = (6432, 10637)$
$p_7 = (18577, 16451)$	$p_{15} = (8912, 17261)$	$p_{23} = (9419, 15893)$	
$p_8 = (10391, 16281)$	$p_{16} = (7842, 19232)$	$p_{24} = (9146, 15771)$	

Table 2: Coordinates of point set  $P_{30}$  with 9776 crossings

### 3 Proof of the Theorems

Let  $S_0 = P$  and  $S_{k+1} = Q(S_k)$  for  $k \geq 0$ , where  $Q(S_k)$  is the set given by Lemma 3. By induction and Lemma 3 it follows that

$$\overline{cr}(S_k) = 16^k \overline{cr}(P) + N^3 8^{k-1} (2^k - 1) - \frac{7}{6} N^2 4^{k-1} (4^k - 1) + \frac{5}{14} N 2^{k-1} (8^k - 1).$$

Letting  $n = |S_k| = 2^k N$  we get

$$\overline{cr}(n) \leq \overline{cr}(S_k) = \left( \frac{24cr(P) + 3N^3 - 7N^2 + (30/7)N}{24N^4} \right) n^4 - \frac{1}{8} n^3 + \frac{7}{24} n^2 - \frac{5}{28} n.$$

$a, (a, b)$ means $(p_a, p_a p_b)$			
1, (1, 3)	9, (9, 13)	17, (17, 18)	25, (25, 26)
2, (2, 4)	10, (10, 11)	18, (18, 19)	26, (26, 27)
3, (3, 7)	11, (11, 12)	19, (19, 21)	27, (27, 28)
4, (4, 5)	12, (12, 14)	20, (20, 21)	28, (28, 29)
5, (5, 8)	13, (13, 14)	21, (21, 23)	29, (29, 27)
6, (6, 7)	14, (14, 15)	22, (22, 23)	30, (30, 29)
7, (7, 9)	15, (15, 16)	23, (23, 24)	
8, (8, 10)	16, (16, 20)	24, (24, 25)	

Table 3: Halving-line matching of  $P_{30}$

This proves Theorem 2. Now, to prove Theorem 1, consider the set  $P = P_{30}$  with coordinates in Table 2 obtained from [2]. It satisfies that  $\overline{cr}(P_{30}) = 9726$  and it can be verified that the set of point-line pairs in Table 3 represents a halving-line matching of  $P_{30}$ . We then get

$$\overline{cr}(n) \leq \frac{29969}{78750} \binom{n}{4} + \Theta(n^3).$$

## References

- [1] Ábrego, B., Fernández-Merchant, S.: A Lower Bound for the Rectilinear Crossing Number, *Graphs and Combinatorics*, to appear (2005).
- [2] Aichholzer, O.: Rectilinear Crossing Number Page. <http://www.ist.tugraz.at/staff/aichholzer/crossings.html>.
- [3] Aichholzer, O., Aurenhammer, F., Krasser, H.: On the crossing number of complete graphs. In: *Proc. 18th Ann ACM Symp Comp Geom.*, pp. 19-24, Barcelona Spain, 2002.
- [4] Aichholzer, O., Krasser, H.: Abstract order type extensions and new results on the rectilinear crossing number. Submitted.
- [5] Balogh, J., Salazar G.: On  $k$ -sets, quadrilaterals and the rectilinear crossing number of  $K_n$ . Manuscript, submitted.
- [6] Brodsky, A., Durocher, S., Gethner, E.: Toward the rectilinear crossing number of  $K_n$ : new drawings, upper bounds, and asymptotics. *Discrete Mathematics* **262**, 59-77 (2003)
- [7] Jensen, H.F.: An upper bound for the rectilinear crossing number of the complete graph. *J. Combin. Theory Ser B* **10**, 212-216 (1971)
- [8] Lovász, L., Vesztegombi, K., Wagner, U., Welzl, E.: Convex quadrilaterals and  $k$ -sets. In: Pach, J. editor: *Towards a theory of geometric graphs*, (Contemporary Mathematics Series, vol. 342, pp.139-148) AMS 2004.
- [9] Matoušek, J.: *Lectures on Discrete Geometry*, New York, N.Y.: Springer-Verlag 2002.
- [10] Pach, J., Tóth, G.: Thirteen problems on crossing numbers. *Geombinatorics* **9**, 194-207 (2000)
- [11] Singer, D.: Rectilinear crossing numbers. Manuscript (1971)
- [12] Scheinerman, E. R., Wilf, H. S., The rectilinear crossing number of a complete graph and Sylvester’s “four point problem” of geometric probability, *Amer. Math. Monthly* **101** (1994), 939-943.
- [13] Sylvester, J.J., On a special class of questions in the theory of probabilities, *Birmingham British Assoc.*, Report **35**, 8-9 (1865).
- [14] Wagner, U.: On the rectilinear crossing number of complete graphs. In: *Proc. of the 14th Annual ACM-SIAM Symposium on Discrete Mathematics (SODA)*, pp. 583-588, 2003.
- [15] Zarankiewicz, K.: On a problem of P. Turán concerning graphs. *Fund. Math.* **41**, 137-145 (1954)