Convex polyhedra in $\mathbb{R}^3$ spanning $(n^{4/3})$ congruent triangles.

Bernardo M. Ábrego
California State University Northridge
bernardo.abrego@csun.edu

Silvia Fernández-Merchant
California State University Northridge
silvia.fernandez@csun.edu

Abstract

We construct n-vertex convex polyhedra with the property stated in the title

In this note we construct, for every fixed triangle T, a n-vertex convex polyhedron determining - $(n^{4/3})$ triangles congruent to T among its triplets. Even with the convexity assumption dropped, this was only known when T is an isosceles right triangle (see [2] and [4]). With respect to the upper bound Brass [2] proved that n points in $\mathbb{R}^3$ span at most $O(n^{7/4+\varepsilon})$ triangles congruent to T and very recently Agarwal and Sharir [1] improved this and obtained the current best bound of $O(n^{5/3+\varepsilon})$. There are no better bounds that take advantage of the convexity restriction.

We say that a finite subset of $\mathbb{R}^3$ is in convex position if it is the vertex set of a convex polyhedron. $CH(K)$ will denote the convex hull of K and ∂K the boundary of K. We prove a slighthly stronger statement. Let U be a quadrilateral with perpendicular diagonals. Assume $u_1 = (d_1, 0, 0), u_2 = (0, 0, d_2), u_3 = (-d_3, 0, 0)$, and $u_4 = (0, 0, -d_4)$ with $d_i > 0$ are the vertices of U and $o = (0, 0, 0)$ is the intersection of its diagonals. For any finite set $P \subseteq \mathbb{R}^3$ let $F(U; P)$ be the number of quadruplets in P congruent to U. Let

$$F_3^{conv}(U; n) = \max \left\{ F(U; P) : P \subseteq \mathbb{R}^3, P \text{ in convex position}, |P| = n \right\},$$

since any triangle T can be completed to such a quadrilateral U (by reflecting upon the largest side), it is enough to prove that

Theorem 1 $F_3^{conv}(U; n) =$ - $(n^{4/3})$

The proof of the Theorem will be based on two lemmas.

For every $0 < \alpha < \frac{\pi}{2}$ and $1 \leq i \leq 4$ define the following arcs of circle

$$Arc_i(\alpha) = \{v = (x, y, z) : y = 0, \|v\| = d_i, |\measuredangle vou_i| < \alpha\}$$

Lemma 1 *There is* $\alpha > 0$ *so that* $\bigcup_{i=1}^{4} Arc_i(\alpha) \subseteq \partial \left(CH \left(\bigcup_{i=1}^{4} Arc_i(\alpha) \right) \right)$.

Proof. Suppose $d_1 \leq d_2$, let $\alpha_{1,2} = \frac{1}{2} \arcsin(d_1/d_2)$. Let a and b be points in the plane $y = 0$ defined by $\|a\| = d_1$,$\|b\| = d_2$, and $\measuredangle u_1oa = \measuredangle bou_2 = \alpha_{1,2}$. By construction $\measuredangle aob = \frac{\pi}{2} - 2\alpha_{1,2}$, thus $\cos(\measuredangle aob) = \sin(2\alpha_{1,2}) = d_1/d_2$, and then $\measuredangle oab = \frac{\pi}{2}$. This proves that $Arc_1(\alpha_{1,2}) \cup Arc_2(\alpha_{1,2}) \subseteq \partial (CH (Arc_1(\alpha_{1,2}) \cup Arc_2(\alpha_{1,2})))$. Clearly any value smaller than $\alpha_{1,2}$ would work for the pair (d_1, d_2), therefore by picking $\alpha \leq \frac{1}{2} \arcsin(\min_{1 \leq i,j \leq 4} d_i/d_j)$ the result follows. ■

Let $e_1 = (1,0,0)$, $e_2 = (0,0,1)$, $e_3 = (-1,0,0)$, $e_4 = (0,0,-1)$, and $S = \{v \in \mathbb{R}^3 : \|v\| = 1\}$. The next lemma without the additional property (ii) was first proved in [4] by Erdős et al.

Lemma 2 *For every $\varepsilon > 0$ and $n \in \mathbb{N}$ there are n-sets $Q_1, Q_2, Q_3, Q_4 \subseteq S$ with the following properties*

(i) There are $cn^{4/3}$ quadruplets (q_1, q_2, q_3, q_4) with $q_i \in Q_i$ and $q_1q_2q_3q_4$ a square of diameter 2.

(ii) $\measuredangle q_i o e_i < \varepsilon$ for every $q_i \in Q_i$, $1 \le i \le 4$.

Proof. Erdős (see [3]) constructed an n-element set P in the plane and a set of n lines L such that the number of incidences among them is at least $cn^{4/3}$ (The set P consists of a $\sqrt{n} \times \sqrt{n}$ grid, and L includes the n lines with more points in P). We can assume that all lines in L have slope smaller than -1 and also that $P \subseteq \{(x, y, -1) \in \mathbb{R}^3 : x \in (m, m+1), y \in (0,1)\}$. For every $p = (x_p, y_p, -1) \in P$ let q_p^1 and q_p^3 be the points obtained as the intersection of S with the line po, i.e., if $p = (x, y, -1)$ then $q_p^1 = -q_p^3 = \|p\|^{-1}(x_p, y_p, -1)$. Also for every $l \in L$ with equation $z = -1$, $A_l x + B_l y = C_l$, ($C_l > 0$ and $A_l^2 + B_l^2 + C_l^2 = 1$) consider the plane π_l through o which contains l. Let q_l^2 and q_l^4 be the points obtained as the intersection of S with the line through o perpendicular to π_l, i.e., $q_l^2 = -q_l^4 = (A_l, B_l, C_l)$.

For $i = 1, 3$ let $Q_i = \{q_p^i : p \in P\}$ and $Q_{i+1} = \{q_l^{i+1} : l \in L\}$. Assume $p \in l$, by construction, q_l^2 and q_l^4 are at distance $\sqrt{2}$ from every point in the circle $\pi_l \cap S$, in particular from q_p^1 and q_p^3. Since q_p^1, q_p^3 and q_l^2, q_l^4 are antipodes on S we conclude that $q_p^1 q_l^2 q_p^3 q_l^4$ is a square of diagonal 2. Therefore the number of such squares in $\bigcup_{i=1}^4 Q_i$ is at least $cn^{4/3}$.

Now, to prove property (ii) we show that for all $p \in P$, $l \in L$, and $i = 1, 3$

$$\lim_{m \to \infty} \left\| q_p^i - e_i \right\| = \lim_{m \to \infty} \left\| q_l^{i+1} - e_{i+1} \right\| = 0.$$

By symmetry we only prove this equality for $i = 1$. If $p \in P$ then $x_p \in (m, m+1)$ and $y_p < 1$, thus

$$\left\| q_p^1 - e_1 \right\|^2 = 2 - \frac{2x_p}{\|p\|} < 2 - \frac{2m}{\sqrt{2 + (m+1)^2}} \longrightarrow 0 \text{ when } m \to \infty.$$

If $l \in L$ then, in the plane $z = -1$, l has slope $-A_l/B_l < -1$ and it intersects the solid square $(m, m+1) \times (0,1)$. Thus $0 < C_l/B_l$ and $m < C_l/A_l$, but since $C_l > 0$ we get $0 < B_l < A_l$ and $A_l < C_l/m$. Hence

$$1 = A_l^2 + B_l^2 + C_l^2 < 2A_l^2 + C_l^2 < C_l^2 \left(\frac{2 + m^2}{m^2} \right),$$

therefore

$$\left\| q_l^2 - e_2 \right\|^2 = 2 - 2C_l < 2 - \frac{2m}{\sqrt{2 + m^2}} \longrightarrow 0 \text{ when } m \to \infty.$$

■

Proof of Theorem. By Lemma 1 there is $0 < \alpha < \pi/2$ so that $\bigcup_{i=1}^{4} Arc_i(\alpha) \subseteq \partial\left(CH\left(\bigcup_{i=1}^{4} Arc_i(\alpha)\right)\right)$. Let $\varepsilon = \alpha$ and apply Lemma 2. For $1 \le i \le 4$ define $P_i = \{d_i q_i : q_i \in Q_i\}$. We claim that $P^* := \bigcup_{i=1}^{4} P_i$ gives the desired bound.

Let $K = CH\left(\bigcup_{i=1}^{4} Arc_i(\alpha)\right)$. Construct K' and K'' as the solids of revolution obtained from K by revolving around the x-axis and the z-axis respectively. Let $K^* = K' \cap K''$, clearly K^* is a convex set, and for $1 \le i \le 4$ the sets $Cap_i(\alpha) = \left\{v \in \mathbb{R}^3 : \|v\| = d_i, |\measuredangle v o u_i| < \alpha\right\}$ are caps of sphere which satisfy that

$$Cap_i(\alpha) \subseteq K' \cap K'' \cap (\partial K' \cup \partial K'') = \partial(K' \cap K'') = \partial K^*.$$

Now, by property (ii) $\measuredangle p_i o u_i < \alpha$ for every $p_i \in P_i$, thus $P_i \subseteq Cap_i(\alpha)$ and $P^* \subseteq \partial K^*$. Finally, since any supporting plane of K^* intersects $\bigcup_{i=1}^{4} Cap_i(\alpha)$ in at most one point, we conclude that P^* is in convex position; and clearly $d_1q_1, d_2q_2, d_3q_3, d_4q_4$ is congruent to U whenever $q_1q_2q_3q_4$ is a square of diameter 2. Hence $F(U; 4n) \ge F(U; P^*) \ge cn^{4/3}$ as we wanted to prove. ■

References

[1] P. Agarwal and M. Sharir (2001), *On the number of congruent simplices in a point set*, in Proc. 17th Annual Symposium on Computational Geometry, ACM Press, New York, 1-9.

[2] P. Brass (2000), *Exact point pattern matching and the number of congruent triangles in a three-dimensional pointset*, in ESA 2000 - European Symposium on Algorithms (M. Paterson, ed.), Lecture Notes in Computer Science 1879, Springer-Verlag, 112-119.

[3] H. Edelsbrunner (1987), *Algorithms in Combinatorial Geometry*, Springer-Verlag, New York.

[4] P. Erdős, D. Hickerson, and J. Pach (1989), A problem of Leo Moser about repeated distances on the sphere, *American Mathematical Monthly* 96, 569-575.