Thu., Aug 5, 2010 **MATH 462, Midterm 2** Prof. V. Panferov

Name: (print) ____________________

CSUN ID No. : Solutions.

This test includes 6 questions (42 points in total), on 6 pages. The duration of the test is 50 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (4 points) Let T be a linear operator on a vector space V. Prove that $N(T)$ and $R(T)$ are T-invariant.

Def: A subspace $W \subseteq V$ is T-invariant if $\forall x \in W$ $T(x) \in W$

$\forall x \in N(T)$ $T(x) = 0 \in N(T)$, since $T(0) = 0$ for any linear operator.

$\forall x \in R(T)$ $T(x) \in R(T)$ by def of the range (inverse image of $T(x)$ is non-empty, contains x as an element.)

2. (8 points) Let T be a linear operator on a finite-dimensional vector space V and let W be the cyclic subspace generated by a nonzero vector $v \in V$. If $k = \dim(W)$ prove that the set $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis of W.

Since $v \neq 0$, $\{v\}$ is lin. independent.
Add vectors $T(v)$, $T^2(v)$ .. to obtain a maximal lin. indep. subset of the form
$S = \{v, T(v), \ldots, T^{\ell-1}(v)\}$.
Claim that $\ell = k$, and the above set is a basis. Indeed, by def. of S,

$$T^{\ell}(v) = a_0 v + a_1 T(v) + \ldots + a_{\ell-1} T^{\ell-1}(v)$$

$$T^{\ell+1}(v) = a_0 T(v) + a_1 T^2(v) + \ldots + a_{\ell-2} T^{\ell-1}(v) + a_{\ell-1}(a_0 v + a_1 T(v) + \ldots + a_{\ell-1} T^{\ell-1}(v))$$

By induction, $\forall m$ (positive integer)

$$T^m(v) = a_0^m v + a_1^m T(v) + \ldots + a_{\ell-1}^m T^{\ell-1}(v).$$

Since $W = \text{span}\{v, T(v), \ldots\}$
it follows that S spans W.
Since by construction S is linearly indep, it is then a basis of W, and $\ell = k = \dim(W)$.

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3. (8 points) Let V be the real vector space of functions spanned by $\{1, t, e^t, te^t, t^2e^t\}$ and let T be the linear operator on V defined by $T(f(t)) = f'(t)$. Find a Jordan basis and a Jordan canonical form of T.

$T(1) = 0 \qquad T(e^t) = e^t \qquad T(t^2e^t) = 2te^t + t^2e^t.$

$T(t) = 1 \qquad T(te^t) = te^t + e^t$

The matrix of T in the given basis is

$$[T]_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Char. poly. is $f(t) = t^2(1-t)^3$.

Eigenvalues are $\lambda_1 = 0$ (multiplicity 2)

and $\lambda_2 = 1$ (multiplicity 3).

Eigenvectors are

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda_1 = 0$$

and

$$w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda_2 = 1.$$

We have $\dim K_{\lambda_1} = 2$, $\dim (K_{\lambda_2}) = 3$.

The Jordan matrix is

$$J = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

All columns in the matrix $[T]_\alpha$ are fine except the last one. To fix the last column choose $v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$

Jordan basis

$\beta = (1, t, e^t, te^t, t^2e^t)$.

Continued...

4. (6 points) Let $\delta : M_{n\times n}(F) \to F$ be a multilinear alternating function of rows of a matrix. Give a direct proof (without using properties of determinants) that if $A \in M_{n\times n}(F)$ is such that $\operatorname{rank}(A) < n$ then $\delta(A) = 0$.

Def: $\delta: M_{n\times n}(F) \to F$ is a multilinear, alternating function of rows if

$$\delta\begin{pmatrix} r_1 \\ \vdots \\ r_i + a r_i' \\ \vdots \\ r_n \end{pmatrix} = \delta\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + a\delta\begin{pmatrix} r_1 \\ \vdots \\ r_i' \\ \vdots \\ r_n \end{pmatrix}$$

and $\delta = 0$ for any matrix with 2 identical rows.

Elementary row operations:

- If B is obtained from A by swapping any 2 rows then $\delta(B) = -\delta(A)$.
- If B is obtained from A by factoring out scalar k from any row, $k\delta(B) = \delta(A)$.
- If B is obtained from A by adding a multiple of any one row to another row then $\delta(B) = \delta(A)$.

From these properties, if B is obtained from A by Gauss reduction with k row swaps and factoring out $p_1 \cdots p_e$ from the pivot rows, then

$$\delta(A) = (-1)^k p_1 \cdots p_e \, \delta(B).$$

If $\operatorname{rank}(A) < n$ then B has at least one zero row $\Rightarrow \delta(B) = 0 \Rightarrow \delta(A) = 0$.

5. (8 points) For the linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_A(x) = Ax$, $x \in \mathbb{R}^n$, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

give an example of a one-dimensional and a two-dimensional invariant subspace. [Hint: it may be helpful to compute the matrix A^2.]

If v is an eigenvector then $\operatorname{span}\{v\}$ is always a one-dimensional invariant subsp.

Char. poly.

$$f(t) = \begin{vmatrix} -t & 0 & 1 \\ 1 & -t & -1 \\ 0 & 1 & 1-t \end{vmatrix} = t^2(1-t) + 1 - t = (t^2+1)(1-t)$$

$\lambda = 1$ – eigenvalue;

$$A - \lambda I = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ – eigenvector.}$$

$E_\lambda = \operatorname{span}\{v\}$ – one-dimensional inv. subspace.

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Char. poly. $g(t) = \begin{vmatrix} -t & 1 & 1 \\ 0 & -1-t & 0 \\ 1 & 1 & -t \end{vmatrix} = -t^2(1+t) + (1+t)$
$= (1-t^2)(1+t) = (1-t)(1+t)^2$.

Eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$

Eigenvector for $\lambda_2 = -1$:

$$A^2 - \lambda_2 I = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \text{ is } w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ (only one)}$$

Now, $Aw_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = w_2$; $Aw_2 = A^2 w_1 = -w_1$

Thus, A has a 2-dimensional cyclic subspace

$$C(w_1) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

Continued...

6. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

Notation F is used for a field of scalars.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) If W_i, $i = 1 \dots n$, are subspaces of a vector space V such that any vector $x \in V$ can be represented uniquely as $x = x_1 + \cdots + x_n$, where $x_i \in W_i$ then $V = W_1 \oplus \cdots \oplus W_n$.

(c) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

(d) There exists a linear operator T with no T-invariant subspace.

(e) The function $\delta : M_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for every $A \in M_{n\times n}(F)$ is an alternating n-linear function.

(f) If $A, B \in M_{n\times n}(F)$ are similar matrices then $\det(A) = \det(B)$.

(g) Any polynomial of degree n with leading coefficient $(-1)^n$ is the characteristic polynomial of some linear operator.

(h) Any linear operator on a finite-dimensional vector space has a Jordan canonical form.

Answers:

(a) F ((I_n is diagonalizable)

(b) T (Thm 5.10c)

(c) T (Thm 5.1)

(d) F ($\{0\}$ and V are always invariant)

(e) T (definition)

(f) T ($\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \det(A)$)

(g) T

(h) F ($f(t)$ has to split)

$f(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + (-1)^n t^n$

is the char. polynomial for

$$A = (-1)^{k+1} \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_0 \\ 1 & & & & a_1 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & a_{n-2} \\ & \cdots & 0 & 1 & a_{n-1} \end{pmatrix}$$

The end.