Tue, Aug 12, 2014 MATH 280, Quiz 7 Prof. V. Panferov

Name (print): Solutions.

Show all your work.

1. (3 points) Use the power series to solve:

$$(x-1)y'' - xy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\begin{array}{r|l} 1 & y = \sum_{n=0}^{\infty} c_n x^n \\ -x & y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \\ (x-1) & y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \end{array}$$

$+$

$$(x-1)y''-xy'+y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{k=1}^{\infty} (k+1)k\, c_{k+1} x^k - \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$(-2c_2 + c_0) + \sum_{k=1}^{\infty} \left[(k+1)k\, c_{k+1} - (k+2)(k+1) c_{k+2} - k c_k + c_k\right] x^k = 0$$

$$-2c_2 + c_0 = 0$$

$$c_{k+2} = \frac{(k+1)k\, c_{k+1} - (k-1)c_k}{(k+2)(k+1)} = \frac{k}{k+2} c_{k+1} - \frac{(k-1)}{(k+2)(k+1)} c_k$$

$k = 1, 2, 3 \ldots$

$c_0 = y(0) = 1; \quad c_1 = y'(0) = 0$

$c_2 = \frac{1}{2} c_0$

$c_3 = \frac{1}{3} c_2 = \frac{1}{3 \cdot 2} c_0$

$c_4 = \frac{2}{4} c_3 - \frac{1}{3 \cdot 4} c_2 = \frac{2}{3 \cdot 4} c_2 - \frac{1}{3 \cdot 4} c_2 = \frac{1}{4 \cdot 3} \cdot c_2 = \frac{1}{4 \cdot 3 \cdot 2} c_0$

$\vdots$

$c_k = \frac{1}{k!} c_0$

Check: $\frac{1}{(k+2)!} = \frac{k}{(k+2)(k+1)!} - \frac{(k-1)}{(k+2)(k+1)(k!)} = \frac{k-k+1}{(k+2)!}$ ✓

Please turn over...

$$y(x) = 1 + \sum_{k=2}^{\infty} \frac{1}{k!} x^k = e^x - x.$$

2. (1 point) Determine the singular points of the given differential equation. Classify each singular point as regular or irregular:

$$\underbrace{(x^2+x-6)}_{(x+3)(x-2)}y'' + (x+3)y' + (x-2)y = 0.$$

$$y'' + \frac{1}{x-2}y' + \frac{1}{x+3}y = 0$$

$x=2$, $x=-3$ - Singular points.

$x=2$: $p(x) = 1$

$q(x) = \frac{(x-2)^2}{x+3}$

analytic at $x=2$

— regular singular pt.

$x=-3$

$p(x) = \frac{x+3}{x-2}$

$q(x) = x+3$

- analytic at $x=-3$

— regular singular pt.

3. (2 points) Find the roots of the indicial equation; write the general form of two linearly independent solutions obtained by Frobenius method. Do not attempt to compute the coefficients.

$$2xy'' - (3+2x)y' + y = 0.$$

$$x^2y'' - \frac{3+2x}{2}xy' + xy = 0$$

$$p(x) = -\frac{3}{2} - x \;;\; p_0 = -\frac{3}{2}$$

$$q(x) = x \;;\; q_0 = 0$$

$$r(r-1) - \frac{3}{2}r = 0$$

$$r\left(r - \frac{5}{2}\right) = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{5}{2}.$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n \;;\; y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{5}{2}}$$

$$= c_0 + c_1x + c_2x^2 + \dots \qquad = \sqrt{x}\,(b_0x^2 + b_1x^3 + b_2x^4 + \dots)$$