Wed, Aug 13, 2014 **MATH 280, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

This test includes 8 questions (total of 38 points + 6 points bonus), on 8 pages. The duration of the test is 1 hour 5 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (4 points) Verify that the functions $y_1(x) = e^x$, $y_2(x) = e^{-x}$ form a fundamental set of solutions of the differential equation:

$$y'' - y = 0.$$

$y_1' = e^x$ $\qquad$ $y_2' = -e^{-x}$

$y_1'' = e^x$ $\qquad$ $y_2'' = e^{-x}$

$y_1'' = y_1$ ✓ $\qquad$ $y_2'' = y_2$ ✓

Compute the Wronskian:

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

Solutions form a fundamental set.

2. (4 points) Solve by variation of parameters:

$$y'' - y = \frac{1}{1+e^x}.$$

(You may refer to the solution of the previous problem.)

$$y(x) = u_1(x)e^x + u_2(x)e^{-x}$$

$$u_1'(x) = \frac{W_1}{W} = \frac{1}{(-2)}\begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = \frac{1}{2}\frac{e^{-x}}{1+e^x}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{1}{(-2)}\begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{1+e^x} \end{vmatrix} = -\frac{1}{2}\frac{e^x}{1+e^x}$$

$$u_1(x) = \frac{1}{2}\int \frac{e^{-x}}{1+e^x}\,dx = \frac{1}{2}\int \frac{dx}{e^x(1+e^x)}$$

$$= \frac{1}{2}\int\left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= -\frac{1}{2}e^{-x} - \frac{1}{2}\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$= -\frac{1}{2}e^{-x} + \frac{1}{2}\int \frac{de^{-x}}{1+e^{-x}}$$

$$= -\frac{1}{2}e^{-x} + \frac{1}{2}\ln(1+e^{-x}) + C_1$$

$$u_2(x) = -\frac{1}{2}\int \frac{e^x}{1+e^x}\,dx = -\frac{1}{2}\int \frac{de^x}{1+e^x}$$

$$= -\frac{1}{2}\ln(1+e^x) + C_2$$

$$y(x) = -\frac{1}{2} + \frac{1}{2}\ln(1+e^{-x})e^x - \frac{1}{2}\ln(1+e^x)e^{-x}$$
$$+ C_1e^x + C_2e^{-x}.$$

Continued...

3. (8 points) Solve the initial-value problem for the Cauchy-Euler equation:

$$x^2y'' + xy' + y = 0, \quad y(1) = 1, \quad y'(1) = 2.$$

$$m(m-1) + m + 1 = 0$$

$$m^2 + 1 = 0$$

$$m = \pm i \quad ;$$

Use
$$x^i = e^{i \ln x}$$
$$= \cos \ln x + i \sin \ln x$$

$$y(x) = c_1 \cos \ln x + c_2 \sin \ln x$$

$$y(1) = c_1 = 1$$

$$y'(x) = -c_1 \frac{1}{x} \sin \ln x + c_2 \frac{1}{x} \cos \ln x$$

$$y'(1) = c_2 = 2$$

$$y(x) = \cos \ln x + 2 \sin \ln x. \qquad (x > 0).$$

4. (6 points) (a) Determine the first (lowest, positive) eigenvalue λ for the problem

$$8\,y'' + \lambda\,y = 0$$
$$y(0) = 0, \quad y(2\pi) = 0.$$

(b) Find the corresponding eigenfunction $y(x)$. (c) Explain the physical significance of the value λ and the eigenfunction $y(x)$ in the problem of buckling of a thin vertical column.

$$y'' + \frac{\lambda}{8} y = 0 \quad ; \quad \lambda > 0$$

$$m^2 + \frac{\lambda}{8} = 0$$

$$m = \pm i\sqrt{\frac{\lambda}{8}}$$

$$y(x) = C_1 \cos\left(\sqrt{\frac{\lambda}{8}}\, x\right) + C_2 \sin\left(\sqrt{\frac{\lambda}{8}}\, x\right)$$

$$y(0) = C_1 = 0$$

$$y(2\pi) = 0 + C_2 \sin\left(\sqrt{\frac{\lambda}{8}} \cdot 2\pi\right) = 0$$

$$\sqrt{\frac{\lambda}{8}} \cdot 2\pi = \pi n, \quad n = 1, 2, 3 \ldots$$

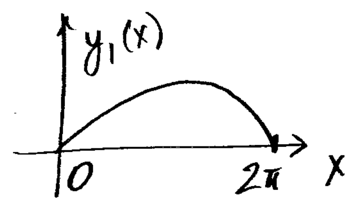
First eigenvalue corresponds to $n = 1$:

$$2\sqrt{\frac{\lambda}{8}} = 1$$

$$\sqrt{\frac{\lambda}{2}} = 1$$

$$\lambda = 2$$

$$y_1(x) = \sin\left(\frac{x}{2}\right)$$



Buckling of a thin vertical column:

$$EI\, y'' + Py = 0$$

$EI = 8$; $P = 2$ - the first critical load (the Euler load)

$y(x)$ - the shape of the first buckling mode.

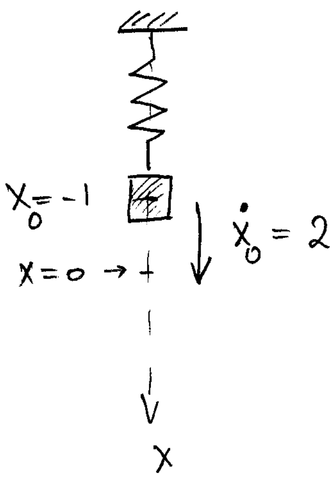
Continued...

5. (6 points) The differential equation of a spring/mass system is

$$\ddot{x} + 16\,x = 0$$

(dot denotes the derivative with respect to t). The mass is initially released from a point 1 meter above the equilibrium position with a downward velocity of 2 m/s.

(a) Find the position of the mass $x(t)$ for $t > 0$.



$\ddot{x} + 16x = 0 \rightsquigarrow m^2 + 16 = 0$

$m = \pm 4i$

$x(0) = -1$, $x'(0) = 2$

$x(t) = c_1 \cos(4x) + c_2 \sin(4x)$

$x'(t) = -4c_1 \sin(4x) + 4c_2 \cos(4x)$

$x(0) = c_1 = -1$

$x'(0) = 4c_2 = 2 \Rightarrow c_2 = \frac{1}{2}$

$x(t) = -\cos(4x) + \frac{1}{2}\sin(4x)$

(positive x- direction of downward.)

(b) Determine the amplitude of the vibrations.

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{1 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$

Continued...

6. (4 points) Consider the equation

$$(x-1)y'' - xy' + y = 0.$$

(a) Determine all singular points and classify them as regular or irregular.

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$$

$$P(x) = -\frac{x}{x-1}; \quad Q(x) = \frac{1}{x-1}$$

— analytic everywhere except $x = 1$.

At $x = 1$,

$$p(x) = (x-1)P(x) = -x; \quad q(x) = (x-1)^2 Q(x) = x-1$$

— analytic at $x = 1$

$\Rightarrow$ $x = 1$ is a regular singular point; all other points are ordinary.

(b) Without solving for the coefficients of the series, determine the minimum radius of convergence of the series solution about $x_0 = 0$.

$x_0 = 0$ — ordinary point

minimal radius of convergence = distance to the closest singular point on the complex plane.

$\Rightarrow R = 1.$

(In fact, the series solution about $x = 0$ converges for all $x \in \mathbb{C}$, See the solution in Quiz 7.)

— the actual radius of convergence is ∞.

7. (6 points) Consider the equation

$$2xy'' - y' + 2y = 0.$$

Given that $r = \frac{3}{2}$ is a root of the indicial equation, find (a) the general form of the corresponding Frobenius series solution; (b) the recurrence relation satisfied by the coefficients of the series. Do not attempt to compute the coefficients.

$$(a) \quad y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+\frac{3}{2}}.$$

$$2. \quad (b) \quad y'(x) = \sum_{n=0}^{\infty} \left(n+\frac{3}{2}\right) c_n x^{n+\frac{3}{2}-1}$$

$$-1.$$

$$2x. \quad y''(x) = \sum_{n=0}^{\infty} \left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) c_n x^{n+\frac{3}{2}-2}$$

$$2xy'' - y' + 2y = 2\sum_{n=0}^{\infty} \left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) c_n x^{n+\frac{3}{2}-1} - \sum_{n=0}^{\infty} \left(n+\frac{3}{2}\right) c_n x^{n+\frac{3}{2}-1}$$

$$+ 2\sum_{n=0}^{\infty} c_n x^{n+\frac{3}{2}}$$

$$= \left(\underbrace{2\cdot\frac{3}{2}\cdot\frac{1}{2} c_0 - \frac{3}{2} c_0}_{=0}\right) + 2\sum_{k=0}^{\infty} \left(k+1+\frac{3}{2}\right)\left(k+1+\frac{1}{2}\right) c_{k+1} x^{k+\frac{3}{2}}$$

$$- \sum_{k=0}^{\infty} \left(k+1+\frac{3}{2}\right) c_{k+1} x^{k+\frac{3}{2}}$$

$$+ 2\sum_{k=0}^{\infty} c_k x^{k+\frac{3}{2}}$$

Recurrence relation:

$$2\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right) c_{k+1} - \left(k+\frac{5}{2}\right) c_{k+1} + 2c_k = 0$$

$$(2k+2)\left(k+\frac{5}{2}\right) c_{k+1} + 2c_k = 0$$

$$c_{k+1} = \frac{-c_k}{(k+1)\left(k+\frac{5}{2}\right)}. \qquad k = 0, 1, 2 \ldots$$

Continued...

8. (bonus: 6 points) Consider the equation for a mass/spring system:

$$\ddot{x} + 2\lambda\,\dot{x} + \omega^2 x = F_0 \sin(\gamma t),$$

where $\lambda, \omega > 0$ are fixed.

(a) Explain why pure resonance (solutions $x(t)$ with unbounded amplitude) does not occur in a system with positive damping ($\lambda > 0$).

$\ddot{X} + 2\lambda\dot{X} + \omega^2 X = 0$

$m^2 + 2\lambda m + \omega^2 = 0$

$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

these are either both negative, or complex congugate with negative real part.

For $\lambda > 0$ m is never purely imaginary

$\Rightarrow$

$X_p(t) = A\cos\gamma t + B\sin\gamma t$;

$\Rightarrow X_p$ has finite amplitude (is bounded.)

$\Rightarrow X_h(t)$ is bounded as $t \to \infty$.

(b) Which value of the external frequency γ corresponds to a steady solution with largest possible amplitude?

Find $X_p(t)$:

$\omega^2 \cdot \quad X_p(t) = A\cos\gamma t + B\sin\gamma t$

$2\lambda \cdot \quad \dot{X}_p(t) = -\gamma A\sin\gamma t + \gamma B\cos\gamma t$

$1 \cdot \quad \ddot{X}_p(t) = -\gamma^2 A\cos\gamma t - \gamma^2 B\sin\gamma t$

$-\gamma^2 A + 2\lambda\gamma B + \omega^2 A = 0$

$-\gamma^2 B - 2\lambda\gamma A + \omega^2 B = F_0$

Determinant:

$$D = \begin{vmatrix} \omega^2-\gamma^2 & 2\lambda\gamma \\ -2\lambda\gamma & \omega^2-\gamma^2 \end{vmatrix} = (\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2 > 0$$

$$A = \begin{vmatrix} 0 & 2\lambda\gamma \\ F_0 & \omega^2-\gamma^2 \end{vmatrix} \Big/ D = \frac{-2\lambda\gamma F_0}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}$$

$$B = \begin{vmatrix} \omega^2-\gamma^2 & 0 \\ -2\lambda\gamma & F_0 \end{vmatrix} \Big/ D = \frac{(\omega^2-\gamma^2)F_0}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}$$

The end.

$$\text{Amplitude} = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}}$$

Amplitude $\to$ max

when $(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2 \to$ min

Check for minimum:

$$\left((\omega^2-\gamma^2)^2+4\lambda^2\gamma^2\right)' = 2(\omega^2-\gamma^2)(-2\gamma)+8\lambda^2\gamma$$

$$= 4\gamma\left(-\omega^2+\gamma^2+2\lambda^2\right) = 0$$

$$\Rightarrow \quad -\omega^2+\gamma^2+2\lambda^2=0 \quad \text{when } (\omega^2>2\lambda^2)$$

$$\gamma^2=\omega^2-2\lambda^2$$

$$\gamma=\sqrt{\omega^2-2\lambda^2} \quad (\approx \omega \text{ for small } \lambda)$$

The maximum amplitude is

$$\frac{F_0}{\sqrt{(2\lambda^2)^2+4\lambda^2(\omega^2-2\lambda^2)}} = \frac{F_0}{\sqrt{4\lambda^2(\omega^2-\lambda^2)}}.$$