Thu, Jul 24, 2014 **MATH 280, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

This test includes 7 questions (total of 48 points + 6 points bonus), on 7 pages. The duration of the test is 50 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) Verify that the functions

$$\text{(a)} \quad y = x, \qquad \text{(b)} \quad y = x\ln(-x)$$

are solutions of the differential equation

$$x^2y'' - xy' + y = 0.$$

Determine intervals of existence for each of the two solutions.

$1 \cdot \quad y = x$

$-x \cdot \quad y' = 1$

$x^2 \cdot \quad y'' = 0$

$+$

$x^2y'' - xy' + y = 0 - x + x = 0$

Interval of existence of $y = x$ is $(-\infty, \infty)$

$1 \cdot \quad y = x\ln(-x)$

$-x \cdot \quad y' = \ln(-x) + 1$

$x^2 \cdot \quad y'' = \frac{1}{x}$

$+$

$x^2y'' - xy' + y = x - x\ln(-x) - x + x\ln(-x) = 0$

Interval of existence is $(-\infty, 0)$.

2. (10 points) (a) Solve the equation

$$y' = 3xy^{2/3}$$

(find the general solution and all singular solutions).

$$\frac{dy}{y^{2/3}} = 3x\,dx$$

$$\int y^{-\frac{2}{3}}\,dy = \int 3x\,dx$$

$$3y^{\frac{1}{3}} = \frac{3x^2}{2} + 3C$$

$$y = \left(\frac{x^2}{2} + C\right)^3.$$

Singular solution $\boxed{y = 0}$ was lost with division by $y^{2/3}$.

(b) Supplement the equation from part (a) with initial data $y(x_0) = y_0$. Determine the region the xy-plane such that the resulting initial-value problem has a unique solution through every point (x_0, y_0) in that region.

$$\text{(IVP)} \begin{cases} y' = 3xy^{2/3} \\ y(x_0) = y_0 \end{cases}$$

has unique solution through (x_0, y_0) if $f(x,y) = 3xy^{2/3}$ and $\frac{\partial f}{\partial y}(x,y) = 2xy^{-1/3}$ are continuous at (x_0, y_0)

$f(x,y)$ is continuous everywhere; $\frac{\partial f}{\partial y}$ is continuous if $\boxed{y_0 \neq 0}$.

(c) Show that solutions of the initial-value problem with the data $y(0) = 0$ are non-unique by finding at least 3 different solutions to the equation with these data.

$y(0) = 0$

$y(x) = 0$ – singular solution

$y(x) = \frac{x^6}{8}$ – solution from the family in (a).

$$y(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^6}{8}, & x \geq 0 \end{cases}$$

Continued...

– differentiable at $x = 0$, satisfies $y' = 3xy^{2/3}$ on $(-\infty, \infty)$.

3. (8 points) Solve the non-exact equation by finding an integrating factor:

$$\underbrace{(y^2 + 3x)}_{M}\, dx + \underbrace{xy}_{N}\, dy = 0.$$

$$M_y = 2y \qquad N_x = y$$

$$\frac{M_y - N_x}{N} = \frac{2y - y}{xy} = \frac{1}{x}$$

$$\mu_x = \frac{M_y - N_x}{N}\mu = \frac{1}{x}\mu$$ has solution

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x. \quad \text{(int. factor.)}$$

Multiply by x:

$$\underbrace{(xy^2 + 3x^2)}_{M} dx + \underbrace{x^2 y}_{N}\, dy = 0$$

$$M_y = 2xy \qquad N_x = 2xy \quad \text{— exact.}$$

$$f(x,y) = \int xy^2 + 3x^2\, dx = \frac{1}{2}x^2y^2 + x^3 + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 y + g'(y) = x^2 y \Rightarrow g'(y) = 0.$$

Solution: $f(x,y) = \frac{1}{2}x^2y^2 + x^3 = C.$

4. (8 points) Solve the linear equation:

$$x^2y' + x(x-2)y = e^{-x}, \quad x > 0.$$

Find the solution that satisfies the initial data $y(1) = \frac{1}{e}$.

$$y' + \frac{x^2-2x}{x^2}y = \frac{e^{-x}}{x^2}$$

$$y' + \left(1-\frac{2}{x}\right)y = \frac{e^{-x}}{x^2}$$

$$\mu(x) = e^{\int 1-\frac{2}{x}\,dx} = e^{x-2\ln x} = x^{-2}e^{x}$$

Multiply by $x^{-2}e^{x}$:

$$\left(\frac{e^{x}}{x^2}y\right)' = \frac{1}{x^4}$$

$$\frac{e^{x}}{x^2}y = -\frac{1}{3x^3} + C$$

$$y = -\frac{1}{3x}e^{-x} + Cx^2e^{-x}.$$

$$y(1) = -\frac{1}{3}e^{-1} + Ce^{-1} = e^{-1}$$

$$-\frac{1}{3} + C = 1$$

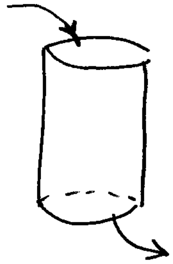
$$C = \frac{1}{3} + 1 = \frac{4}{3}$$

$$y(x) = \left(\frac{4}{3}x^2 - \frac{1}{3x}\right)e^{-x}.$$

5. (8 points) A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the talk at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate.

(a) Set up a differential equation for the amount $A(t)$ of salt in the tank at time t. (Show how the equation is derived.) What are the initial conditions? $\leftarrow A(0) = 30.$

4L/min
1g/L



$V = 200$ L

$A_0 = 30$ g

4L/min

$\frac{A(t)}{V}$ g/L

$$A(t+\Delta t) = A(t) + \Delta t \cdot 4 \cdot 1 - \Delta t \cdot 4 \cdot \frac{A(t)}{200}$$

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} = 4 - \frac{1}{50} A(t)$$

Let $\Delta t \to 0$; $\frac{dA}{dt} = 4 - \frac{1}{50} A$

(b) Solve the equation to find $A(t)$.

$$\frac{dA}{dt} + \frac{1}{50} A = 4$$

$$\mu(t) = e^{\frac{1}{50}t}$$

$$\left(A e^{\frac{1}{50}t}\right)' = 4 e^{\frac{1}{50}t}$$

$$A e^{\frac{1}{50}t} = \int 4 e^{\frac{1}{50}t} dt = 200 e^{\frac{1}{50}t} + C$$

$$A(t) = 200 + Ce^{-\frac{1}{50}t}.$$

Use $A(0) = 30$ to find C:

$30 = 200 + C$

$C = -170$

$A(t) = 200 - 170 e^{-\frac{1}{50}t}.$

(c) What is the concentration of salt in the tank after a long time ($t \to \infty$)?

$$t \to \infty \Rightarrow e^{-\frac{1}{50}t} \to 0$$

$$A(t) = 200 - 170 e^{-\frac{1}{50}t} \longrightarrow 200 \text{ [g]}$$

Limiting concentration is

$$\frac{200 \text{ g}}{200 \text{ L}} = 1 \text{ g/L}.$$

(same as in the inflow.)

Continued...

6. (8 points) Consider the logistic equation:

$$\frac{dP}{dt} = P(1-P).$$

(a) Find all critical points/equilibrium solutions.

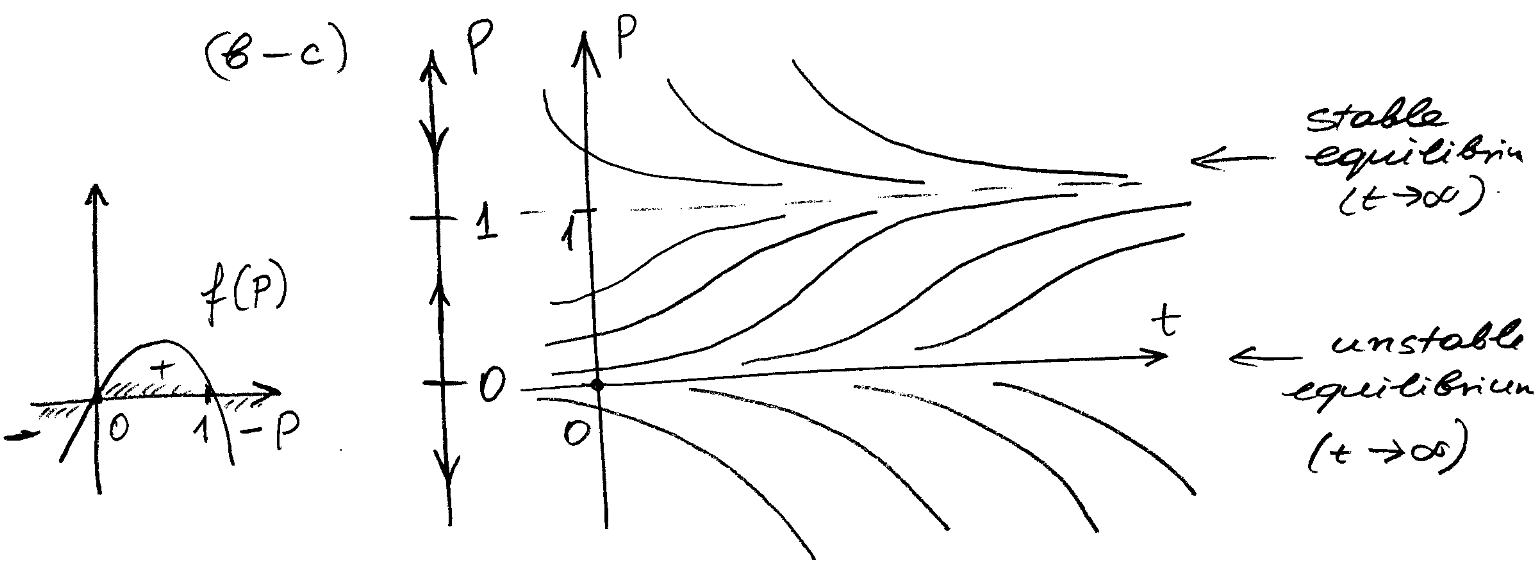
(b) Draw a "phase portrait" showing the intervals of increase/decrease of solutions.

(c) Sketch a graph showing a few typical solution curves.

(a) $f(P) = P(1-P) = 0 \iff P = 0$ or $P = 1$

crit. points

(b–c)



(d) Show, without solving the equation that solution curves have an inflection point when $P = \frac{1}{2}$.

$$\frac{d^2P}{dt^2} = \frac{d}{dP}\left(P - P^2\right)\frac{dP}{dt}$$

$$= (1-2P)P(1-P)$$

$\frac{dP}{dt} = 0$ if $P = \frac{1}{2}$ (also when $P=0$ or $P=1$ but those are equilibrium solutions)

$\Rightarrow P = \frac{1}{2}$ corresponds to inflection pt.

Continued...

7. (bonus: 6 points) Consider the following population model

$$\frac{dP}{dt} = P \ln P, \quad P(0) = P_0 > 1.$$

Solve the equation, and determine which of the following is true:

(a) the model predicts "doomsday" at certain finite time T (*i. e.* solution escapes to $+\infty$ along a vertical asymptote)

or

(b) solution $P(t)$ can be continued for all $t > 0$.

$$\frac{dP}{P \ln P} = dt$$

$$\int \frac{dP}{P \ln P} = \int dt = t + C_1$$

$$\int \frac{dP}{P \ln P} = \int \frac{dv}{v} = \ln v = \ln(\ln P)$$

$$v = \ln P$$

$$\ln(\ln P) = t + \ln(\ln P_0)$$

$$\ln P = (\ln P_0) e^t$$

$$P(t) = e^{(\ln P_0) e^t}$$

Interval of existence is $(-\infty, \infty)$

(but $P(t)$ grows very rapidly...)

Thus, (b) is true.