Mon, Jul 1, 2013 **MATH 250, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

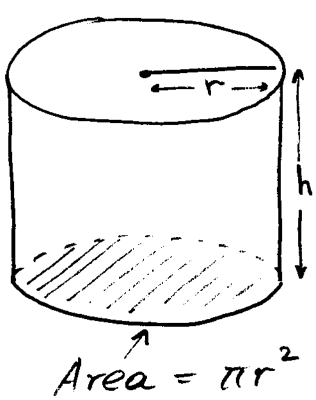
This test includes 7 questions (total of 46 points), on 7 pages. The duration of the test is 60 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) At a certain instant a right circular cylinder has radius of base 10 in. and height 15 in. At this instant the radius is decreasing at the rate of 5 in/sec and the height is increasing at the rate of 4 in/sec. How rapidly is the volume changing at this moment?



$$V = \pi r^2 h$$

$$V'(t) = 2\pi r h\, r'(t) + \pi r^2 h'(t)$$

$$= 2\pi \cdot 10 \cdot 15 \cdot (-5) + \pi \cdot 10^2 \cdot 4$$

$$= \pi\, 10^2 (-15+4) = -1100\pi \ \text{in}^3/\text{s}$$

$$\approx -3455.8 \ \text{in}^3/\text{s}$$

$$\approx -2.0 \ \text{ft}^3/\text{s}$$

(the volume is decreasing.)

2. (6 points) Given the function

$$g(x,y) = \frac{1}{\sqrt{25 - x^2 - y^2}}.$$

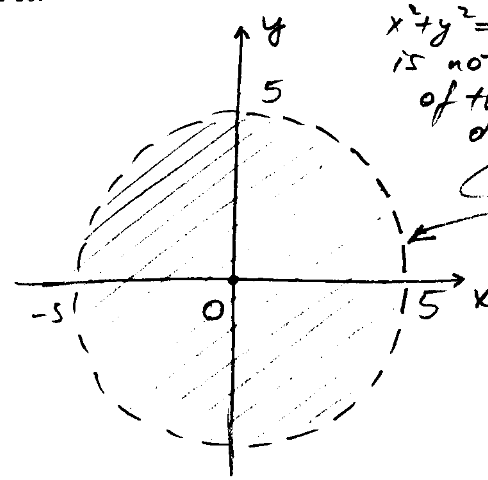
(a) Find the domain D of the function $g(x,y)$ and sketch it.

$25 - x^2 - y^2 > 0$

$\Leftrightarrow \; x^2 + y^2 < 25$

$D = \{(x,y) : \; x^2+y^2 < 25\}$

the open disk of radius 5.



The circle $x^2+y^2=25$ is not part of the domain

(b) Sketch level curves $g(x,y) = k$ for $k = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}$, and $k = 100$.

$k = \frac{1}{5}$: $\sqrt{25 - x^2 - y^2} = 5$

$\Leftrightarrow x^2 + y^2 = 0$

$\Leftrightarrow (x,y) = (0,0)$ - the origin

$k = \frac{1}{4}$: $\sqrt{25 - x^2 - y^2} = 4$

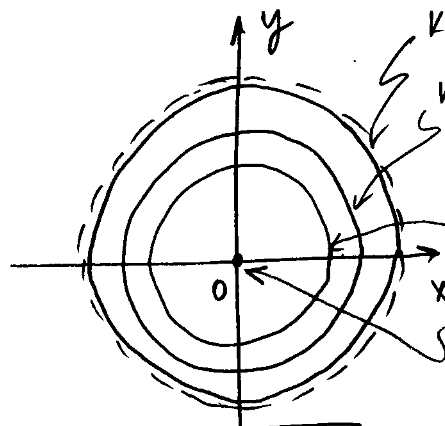
$25 - x^2 - y^2 = 16$

$x^2 + y^2 = 25 - 16 = 9$

- circle of radius 3

$k = \frac{1}{3}$: $\sqrt{25 - x^2 - y^2} = 3$

$x^2 + y^2 = 16$ - circle of radius 4



$k = 100$: $\sqrt{25 - x^2 - y^2} = \frac{1}{100}$

$x^2 + y^2 = 25 - \frac{1}{10^4}$ - circle of radius ≈ 5.0

(c) Based on the level curves, what can be said about the global minimum of $g(x,y)$ on D? The global maximum?

The global minimum $f(0,0) = \frac{1}{5}$

There is no global maximum,

$f(x,y) \longrightarrow \infty$, as $x^2+y^2 \rightarrow 25-0$.

3. (6 points) Find the limits, or show that they do not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$

$$f(x,y) = \frac{4xy}{x^2+y^2}$$

$$x = 0 \Rightarrow f(0,y) = 0$$

$$y = 0 \Rightarrow f(x,0) = 0$$

$$x = y \Rightarrow f(x,x) = \frac{4x^2}{2x^2} = 2 \quad (x \neq 0)$$

$f(x,y)$ approaches different values along different paths approaching the origin $\Rightarrow$ the limit <u>does not exist</u>.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-2x^2y^2}{x^2+y^2}$; $\quad f(x,y) = \dfrac{x^2+y^2-2x^2y^2}{x^2+y^2}$.

$$\frac{x^2+y^2-2x^2y^2}{x^2+y^2} = 1 - \frac{2x^2y^2}{x^2+y^2} = 1 - xy\left(\frac{2xy}{x^2+y^2}\right)$$

≤ 1

$-1 \le$

approaches 0 as $(x,y)\to(0,0)$.

Use Squeeze Principle:

since $-|xy| \le xy \le |xy|$, $\quad -1 \le \dfrac{2xy}{x^2+y^2} \le 1$,

$$\underbrace{1-|xy|}_{\downarrow\, (x,y)\to(0,0)} \le 1 - xy\,\frac{2xy}{x^2+y^2} \le \underbrace{1+|xy|}_{\downarrow,\ (x,y)\to(0,0)}$$

$\qquad 1 \qquad\qquad\qquad\qquad\qquad\qquad 1$

Continued...

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 1.$$

4. (6 points) Given the function

$$f(x, y, z) = 3x^2 + y^3 + z^4 - 4xyz \quad \text{and the point} \quad P(1, 1, 1).$$

(a) Find the gradient vector $\vec{\nabla} f$ and determine the gradient vector at the point P.

$$\vec{\nabla} f = (6x - 4yz,\ 3y^2 - 4xz,\ 4z^3 - 4xy)$$

$$\vec{\nabla} f(1,1,1) = (2, -1, 0)$$

(b) Find the directional derivative $\partial_{\vec{u}} f$ at the point P in the direction of the vector $\vec{u} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

$$\partial_{\vec{u}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \vec{u} = (2, -1, 0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

(b) Find the <u>maximum value</u> of the directional derivative $\partial_{\vec{u}} f$ at the point P for all directions $\vec{u}$.

$$|\vec{\nabla} f(1,1,1)| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}.$$

Continued...

5. (6 points) Show that the point $(x_0, y_0) = (-1, -1)$ is a critical point of the function

$$f(x,y) = x^3 - y^3 + 3x^2y - 9x$$

and determine its type (local maximum/minimum, or saddle point).

$$\nabla f = \left(3x^2 + 6xy - 9,\ -3y^2 + 3x^2\right)$$

$$\nabla f(-1,-1) = (3+6-9,\ -3+3) = (0,0)$$

$\Rightarrow (-1,1)$ is a critical point.

$$D^2 f = \begin{pmatrix} 6x+6y & 6x \\ 6x & -6y \end{pmatrix}$$

$$D^2 f(-1,-1) = \begin{pmatrix} -12 & -6 \\ -6 & 6 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\Delta = AC - B^2 = -12 \cdot 6 - 6 \cdot 6 < 0$$

$\Rightarrow (-1,1)$ is a saddle point.

6. (8 points) Use Lagrange's method to find the maximum and minimum of $f(x, y)$, subject to the given constraint:

$$f(x,y) = y^2 - x, \quad \text{constraint: } \overbrace{2x^2 + y^2}^{g(x,y)} = 1.$$

$$\vec{\nabla} f = (-1, 2y) \; ; \quad \vec{\nabla} g = (4x, 2y)$$

Lagrange multiple equations: $\vec{\nabla} f = \lambda \vec{\nabla} g$:

$$\left.\begin{array}{l} -1 = 4\lambda x \\ 2y = 2\lambda y \end{array}\right\} \Rightarrow y = 0 \text{ or } \lambda = 1$$

<u>Case 1</u>: $y = 0 \Rightarrow 2x^2 + 0^2 = 1 \Rightarrow 2x^2 = 1$

$$\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

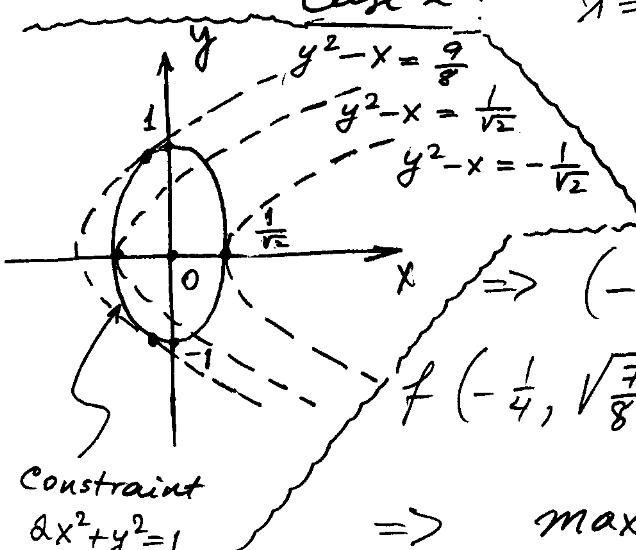
$\Rightarrow \left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)$ - critical points.

$$f\left(\frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{\sqrt{2}}, \quad f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}$$

<u>Case 2</u>: $\lambda = 1 \Rightarrow -1 = 4x \Rightarrow x = -\frac{1}{4}$

$$\Rightarrow y^2 = 1 - 2x^2 = 1 - 2 \cdot \frac{1}{16} = \frac{7}{8}$$

$$\Rightarrow y = \pm \sqrt{\frac{7}{8}}$$



$\Rightarrow \left(-\frac{1}{4}, \sqrt{\frac{7}{8}}\right), \left(-\frac{1}{4}, -\sqrt{\frac{7}{8}}\right)$ - critical pts.

$$f\left(-\frac{1}{4}, \sqrt{\frac{7}{8}}\right) = f\left(-\frac{1}{4}, -\sqrt{\frac{7}{8}}\right) = \frac{7}{8} + \frac{1}{4} = \frac{9}{8}$$

$\Rightarrow$ maximum $f\left(-\frac{1}{4}, \pm\sqrt{\frac{7}{8}}\right) = \frac{9}{8} \quad \left(\frac{9}{8} > 1 > \frac{1}{\sqrt{2}}\right)$

minimum $f\left(\frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{\sqrt{2}}$

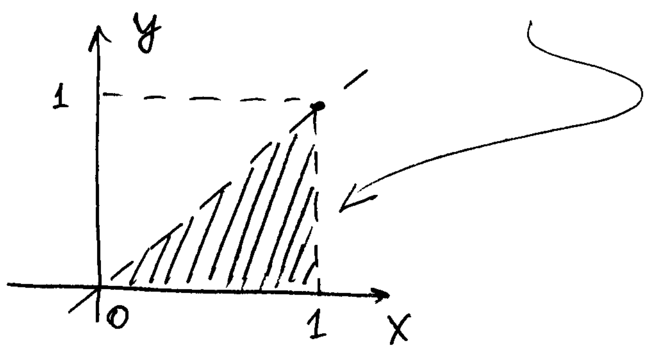
Continued...

7. (8 points) Given the iterated integral:

$$\int_0^1 \int_y^1 \sqrt{1+x^2}\, dx\, dy$$

(a) Sketch the region R over which the integration is performed.

$R = \{(x,y) : 0 \le y \le 1, \; y \le x \le 1\}$



(b) Write the equivalent iterated integral in the reverse order.

$$\int_0^1 \int_0^x \sqrt{1+x^2}\, dy\, dx = \int_0^1 x\sqrt{1+x^2}\, dx$$

(c) Evaluate the integral obtained in (b).

$$\int_0^1 x\sqrt{1+x^2}\, dx = \int_0^1 \sqrt{1+u}\, \frac{1}{2} \cdot du = \frac{1}{2} \cdot \frac{2}{3} (1+u)^{3/2} \Big|_{u=0}^{u=1}$$

$x^2 = u$
$2x\,dx = du$
$x\,dx = \frac{1}{2} du$

$$= \frac{1}{3}\left(2^{3/2} - 1^{3/2}\right)$$

$$= \frac{2\sqrt{2} - 1}{3}$$

$$\approx 0.6095$$

THE END.