Thu, Jun 13, 2013 **MATH 250, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

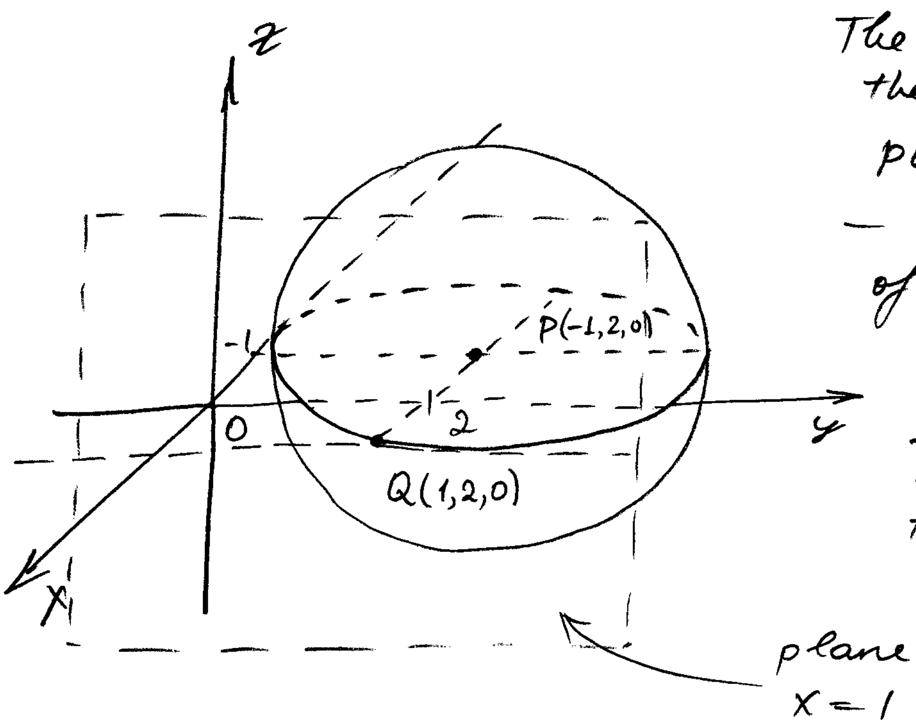
This test includes 8 questions (total of 50 points), on 10 pages. The duration of the test is 60 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (4 points) Find an equation of the sphere that has center at $P(-1,2,0)$ and touches the plane $x = 1$. Determine the point where it touches the plane.



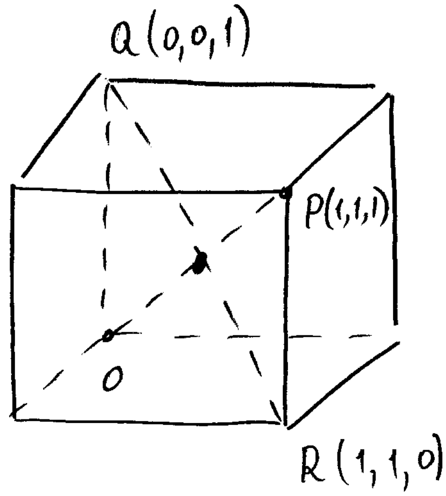
The sphere touches the plane at the point $Q(1,2,0)$ — it is the projection of the center onto the plane $\{x=1\}$.

This determines the radius $|PQ| = 2$

Equation of the sphere:

$$(x+1)^2 + (y-2)^2 + z^2 = 4.$$

2. (4 points) Use the dot product of vectors to find the angle between two diagonals of a cube.



$\vec{OP} = (1,1,1)$

$\vec{QR} = (1,1,-1)$

$$\cos\theta = \frac{\vec{OP}\cdot\vec{QR}}{|\vec{OP}|\,|\vec{QR}|} = \frac{1+1-1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$

$$\theta = \arccos\frac{1}{3} \approx 1.23 \text{ rad} \approx 70.5^\circ.$$

3. (4 points) Find a vector function that represents the curve of intersection of the two surfaces:

The cylinder $x^2 + y^2 = 25$, and the paraboloid $z = xy$.

(x,y) are parameterized as

$x = 5\cos t$

$y = 5\sin t$

Then $z = 5\cos t \cdot 5\sin t = 25\cos t \sin t.$

4. (6 points) Find an equation of the plane through the points $P(3, 1, -1)$, $Q(0, 4, 0)$ and $R(5, 3, 1)$. Check that your points lie on the plane.

$$\vec{PQ} = (-3, 3, 1)$$

$$\vec{PR} = (2, 2, 2)$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 3 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 4\vec{i} + 8\vec{j} - 12\vec{k}$$

= normal vector.

$$= 4(\vec{i} + 2\vec{j} - 3\vec{k})$$

$$= 4(1, 2, -3)$$

Equation of the plane: (point Q used for simplicity)

$$1 \cdot (x-0) + 2(y-4) - 3(z-0) = 0$$

$$\boxed{x + 2y - 3z = 8.}$$

5. (6 points) Consider the line

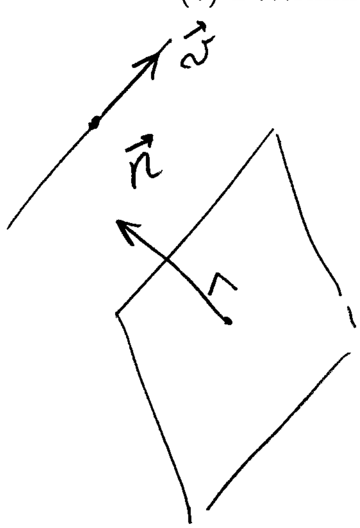
$$\frac{x-2}{3} = \frac{1-y}{2} = 2z+1.$$

(a) Write the equation of the line in <u>vector form</u>.

$$\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+\frac{1}{2}}{\frac{1}{2}}$$

$$(x,y,z) = \left(2,\ 1,\ -\tfrac{1}{2}\right) + t\underbrace{\left(3,\ -2,\ \tfrac{1}{2}\right)}_{\text{direction vector}}$$

(b) Determine whether the line is parallel to the plane $x + 2y + 2z = 6$.



A line is parallel to a plane if the direction vector of the line is perpendicular to the normal vector of the plane.

$$\vec{v} = \left(3,\ -2,\ \tfrac{1}{2}\right)$$

$$\vec{n} = (1,\ 2,\ 2)$$

$$\vec{v}\cdot\vec{n} = 3 - 4 + 1 = 0$$

YES, the line is parallel to the plane.

6. (6 points) Find parametric equations for the tangent line to the curve with the given parametric equations at the given point:

$$\vec{r}(t) = (x, y, z) = \left(\tan t, \ln\cos t, \frac{t}{\sqrt{1+t^2}}\right), \quad t = 0.$$

$$\vec{r}(0) = (0, 0, 0)$$

$$(\tan t)' = \sec^2 t = \frac{1}{\cos^2 t}$$

$$(\ln\cos t)' = \frac{-\sin t}{\cos t} = -\tan t$$

$$\left(\frac{t}{\sqrt{1+t^2}}\right)' = \frac{1}{\sqrt{1+t^2}} - \frac{t \cdot 2t}{2(1+t^2)^{3/2}} =$$

$$= \frac{1+t^2-t^2}{(1+t^2)^{3/2}} = \frac{1}{(1+t^2)^{3/2}}$$

$$\vec{r}\,'(t) = \left(\frac{1}{\cos^2 t}, -\tan t, \frac{1}{(1+t^2)^{3/2}}\right)$$

$$r'(0) = (1, 0, 1)$$

Tangent line:

$$(x, y, z) = (0, 0, 0) + t\,(1, 0, 1)$$

or

$$\begin{cases} x = t \\ y = 0 \\ z = t. \end{cases}$$

7. (8 points) Consider the equation of a quadric surface:

$$9x^2 + 36y^2 - 4z^2 - 36x - 72y + 36 = 0.$$

(a) Reduce the equation to one of the standard forms.

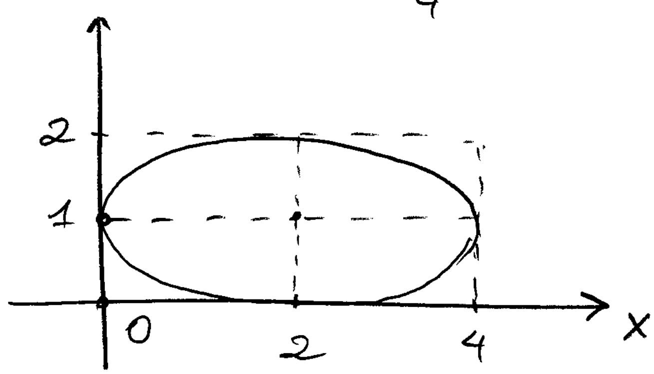
$$9(x^2 - 4x + 4) - 36 + 36(y^2 - 2y + 1) - 36 - 4z^2 + 36 = 0$$

$$9(x-2)^2 + 36(y-1)^2 - 4z^2 = 36$$

$$\frac{(x-2)^2}{4} + (y-1)^2 - \frac{z^2}{9} = 1$$

(b) Find the equation for the section of the surface by the xy-plane and sketch it.

$z = 0 \Rightarrow \quad \frac{(x-2)^2}{4} + (y-1)^2 = 1$ (ellipse)



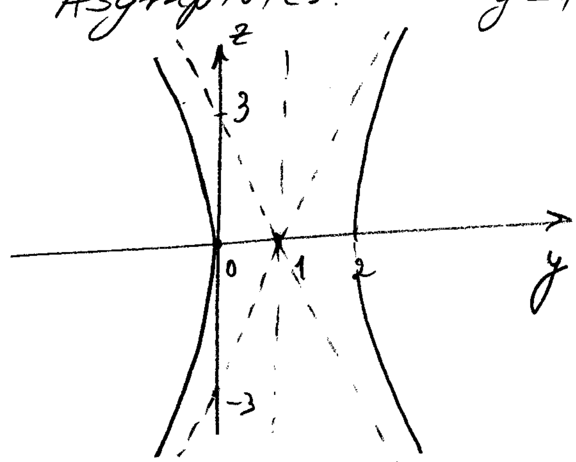
(c) Find the equation for the section of the surface by the plane $x = 2$ and sketch it.

$(y-1)^2 - \frac{z^2}{9} = 1$ (hyperbola)

Asymptotes: $\quad y - 1 = \pm \frac{z}{3}$; $\quad z = \pm 3(y-1)$

$z = 0 \Rightarrow (y-1)^2 = 1$

$\Rightarrow y = 0$ or 2



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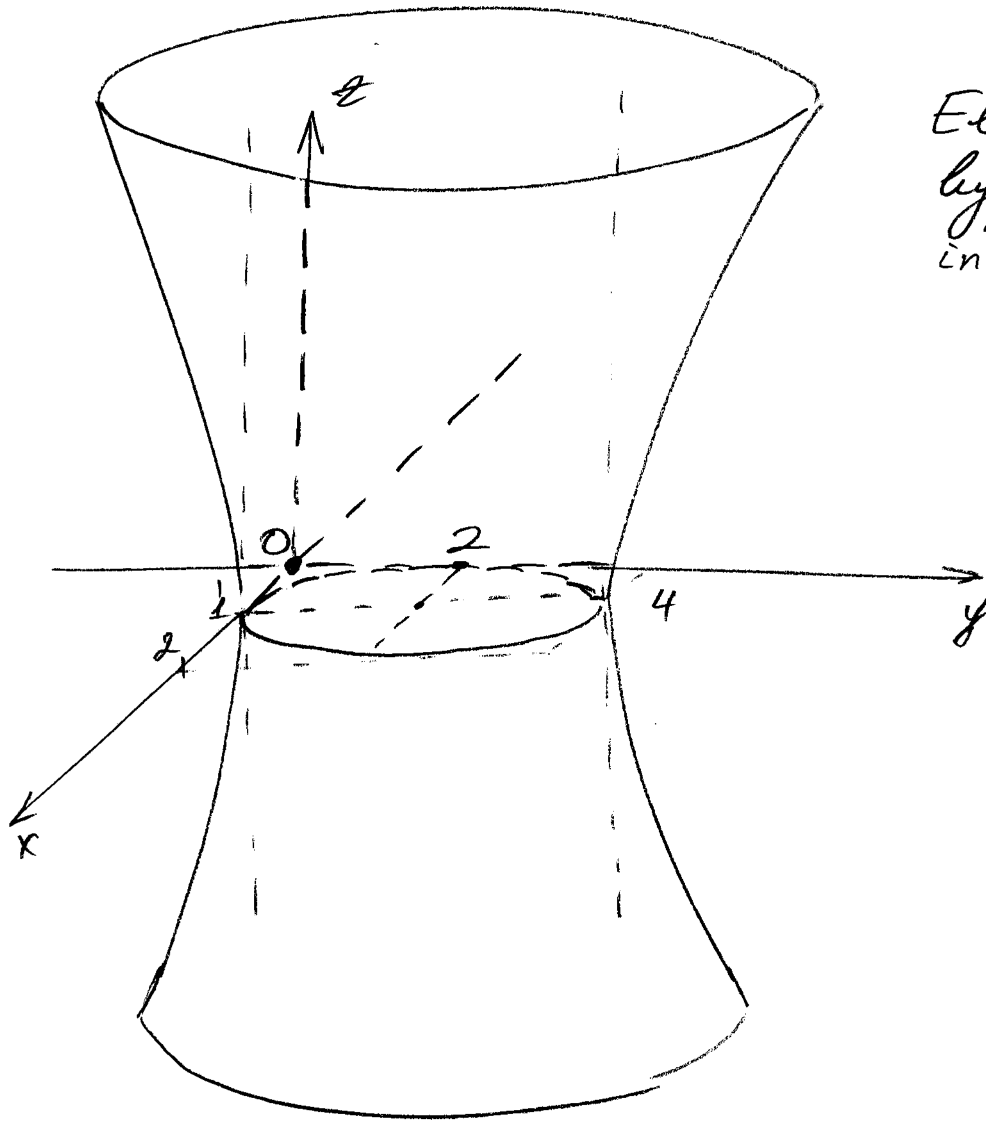
(d) Use the information obtained in parts (a), (b), (c) to make an accurate sketch of the surface. Show the sections by the planes from (b) and (c) on your graph. Classify the surface.



Elliptic hyperboloid in one sheet.

8. (12 points) Consider the curve given by the vector function

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \ln\cos t\,\vec{k}, \quad 0 \le t < \frac{\pi}{2}.$$

(a) Find the derivative $\vec{r}'(t)$ and its magnitude $|\vec{r}'(t)|$.

$$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} - \tan t\,\vec{k}.$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1+\tan^2 t}$$

$$= \sqrt{1+\frac{\sin^2 t}{\cos^2 t}} = \sqrt{\frac{\cos^2 t+\sin^2 t}{\cos^2 t}} = \frac{1}{\cos t}$$

(b) Find the unit tangent vector $\vec{T}(t)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \cos t\left(-\sin t\,\vec{i} + \cos t\,\vec{j} - \tan t\,\vec{k}\right)$$

$$= -\sin t\cos t\,\vec{i} + \cos^2 t\,\vec{j} - \sin t\,\vec{k}$$

$$= -\frac{1}{2}\sin(2t)\,\vec{i} + \cos^2 t\,\vec{j} - \sin t\,\vec{k}.$$

(c) Find the curvature vector $\vec{\kappa}(t)$ and the curvature $\kappa(t)$.

$$\vec{k}(t) = \frac{\vec{T}'(t)}{|\vec{r}'(t)|} = \cos t\left(-\cos 2t\,\vec{i} - 2\cos t\sin t\,\vec{j} - \cos t\,\vec{k}\right)$$

$$= -\cos t\cos 2t\,\vec{i} - \cos t\sin 2t\,\vec{j} - \cos^2 t\,\vec{k}$$

$$\kappa(t) = |\vec{k}(t)| = \cos t\sqrt{\cos^2 2t + \sin^2 2t + \cos^2 t}$$

$$= \cos t\sqrt{1+\cos^2 t}.$$

(d) Find the length of the curve from $t = 0$ to $t = \frac{\pi}{4}$.

$$\ell(C) = \int_0^{\pi/4} |\vec{r}'(t)|\,dt = \int_0^{\pi/4} \frac{1}{\cos t}\,dt$$

$$= \Big[\ln|\sec t + \tan t|\Big]_{t=0}^{t=\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln 1$$

$$= \ln(\sqrt{2}+1) \approx 0.8814.$$

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(e) Show that the curve lies on the cylinder $x^2 + y^2 = 1$.

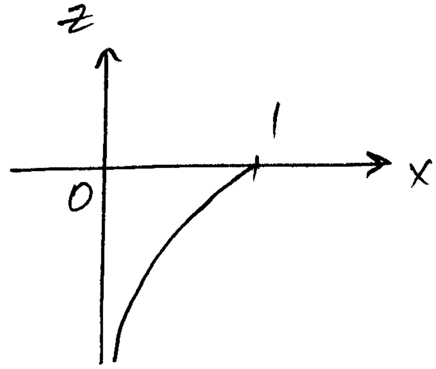
$x = \cos t,\ y = \sin t \Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$

(f) Find the projection of the curve on the (x, z)-plane and sketch it. *Hint: Look for an equation of the form $z = f(x)$. Note that since $x = \cos t$, $0 \le t < \frac{\pi}{2}$, we have $0 \le x < 1$.*

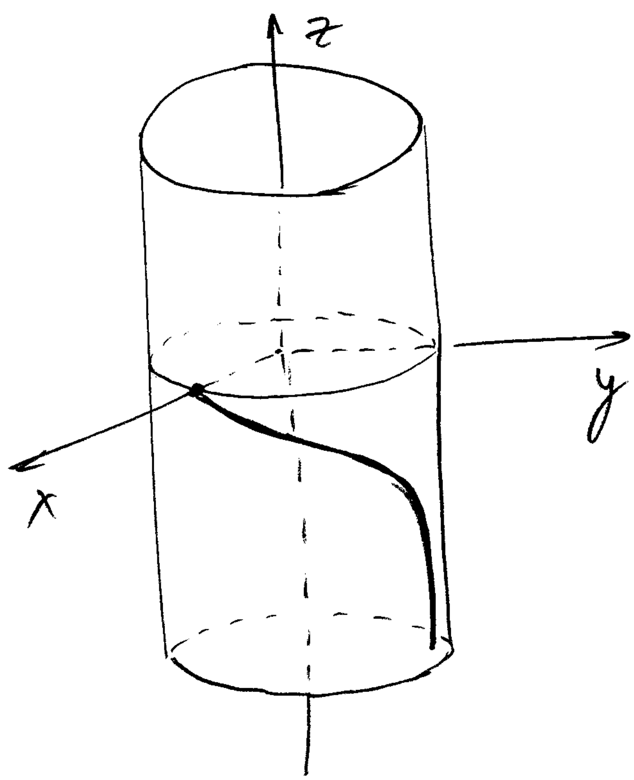
xz-plane:

$x = \cos t$

$z = \ln \cos t = \ln x$



(g) Use the information obtained in parts (e) and (f) to make an accurate sketch of the curve.



Some useful trigonometric integrals

$$\int \sec t\, dt = \ln|\tan t + \sec\ t| + C$$

$$\int \sec^2 t\, dt = \tan t + C$$

$$\int \tan t\, dt = -\ln|\cos t| + C$$

THE END.