#6. (16.3)

$$\vec{F}(x,y) = (3x^2 - 2y^2)\vec{i} + (4xy+3)\vec{j} = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

$$\frac{\partial P}{\partial y} = -4y \;;\quad \frac{\partial Q}{\partial x} = 4y$$

$\Rightarrow$ $\vec{F}(x,y)$ is not conservative.

#8. (16.3)

$$\vec{F}(x,y) = (2xy + y^{-2})\vec{i} + (x^2 - 2xy^{-3})\vec{j}\,,\quad y>0$$

$$= P(x,y)\vec{i} + Q(x,y)\vec{j}$$

$$\frac{\partial P}{\partial y} = 2x - 2y^{-3} \;;\quad \frac{\partial Q}{\partial x} = 2x - 2y^{-3}$$

Try to find $f(x,y)$. $\vec{F} = \vec{\nabla} f$.

$$\frac{\partial f}{\partial x} = 2xy + y^{-2}$$

$$f(x,y) = \int (2xy + y^{-2})\,dx + C(y)$$

$$= x^2y + xy^{-2} + C(y)$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 - 2xy^{-3} + C'(y) \Rightarrow C'(y) = 0$$

$C(y) = C$ (a constant.)

$$f(x,y) = x^2y + xy^{-2} + C.$$

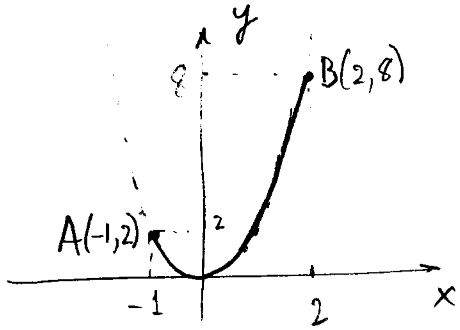
#12. (16.3)

$$\vec{F}(x,y) = x^2\vec{i} + y^2\vec{j}$$

In this case we can guess

$$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3}$$

(the procedure as above works too)



$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

$$= f(2,8) - f(-1,2)$$

$$= \frac{2^3+8^3}{3} - \frac{(-1)^3+2^3}{3}$$

$$= \frac{8^3+1^3}{3} = \frac{513}{3} = 171.$$

#20. $\int_C \sin y \, dx + (x\cos y - \sin y)\, dy$

C: from $A(2,0)$ to $B(1,\pi)$.

Check $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$: $P(x,y) = \sin y$

$Q(x,y) = x\cos y - \sin y$

$\frac{\partial P}{\partial y} = \cos y$; $\frac{\partial Q}{\partial x} = \cos y$.

$\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous on $\mathbb{R}^2$

By Theorem 6 in 16.3 $\vec{F} = (P,Q)$ is conservative

To evaluate the integral, find f : $\vec{F} = \vec{\nabla} f$.

$\frac{\partial f}{\partial x} = \sin y \Rightarrow f(x,y) = x\sin y + C(y)$

$\frac{\partial f}{\partial y} = x\cos y + C'(y) \Rightarrow C'(y) = -\sin y$

$C(y) = \cos y + C$

$f(x,y) = x\sin y + \cos y + C$

$$\int_C P\,dx + Q\,dy = f(B) - f(A) = f(1,\pi) - f(2,0)$$ 3

$$= \underbrace{1 \cdot \sin\pi}_{=0} + \underbrace{\cos\pi}_{=-1} - 2 \cdot \sin 0 - \underbrace{\cos 0}_{=1}$$

$$= -2.$$

#30.
(16.3)

$$\int_C y\,dx + x\,dy + xy\,dz$$

$$\vec{F} = (P,Q,R) = (y, x, xy)$$

$$\frac{\partial P}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = 1 \quad \checkmark$$

$$\frac{\partial P}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = y \quad \times$$

$$\frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial y} = x \quad \times$$

$\vec{F}$ is Not Conservative.

#35. (16.3)

$$\vec{F}(x,y) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2} = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

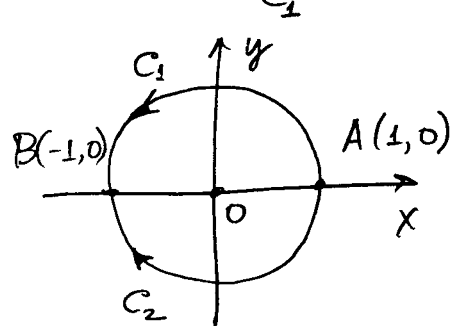
(a)

$$\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

(b)

$$\int_{C_1} \vec{F}\cdot dr = \int_{C_1} P\,dx + Q\,dy$$



$$\left[\begin{array}{l} C_1: \quad x = \cos t,\ y = \sin t, \qquad 0 \le t \le \pi \\ dx = -\sin t\,dt, \quad dy = \cos t\,dt \end{array}\right]$$

$$= \int_0^{\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}\cos t \right) dt$$

$$= \int_0^{\pi} 1\,dt = \pi$$

$$\left[\begin{array}{l} C_2: \quad x = \cos t,\ y = -\sin t, \qquad 0 \le t \le \pi \\ dx = -\sin t,\ dy = -\cos t\,dt \end{array}\right]$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \left(\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(-\cos t) \right) dt$$

$$= \int_0^{\pi} (-1)\,dt = -\pi$$

This is in no contradiction to Theorem 6, since $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are not continuous at $(0,0)$
$\Rightarrow$ one cannot choose a simply connected domain that contains both paths, such that $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous there.