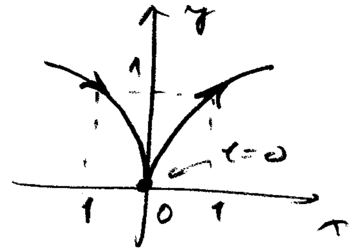
Homework for Sections 13.1 - 13.2 + 13.3 (arc length)

— Solutions.

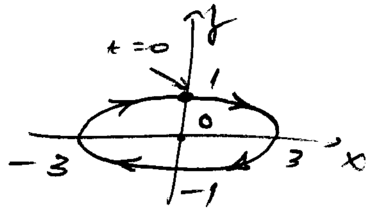
1.

(a) $x = t^3$, $y = t^2 \Rightarrow y = x^{2/3}$



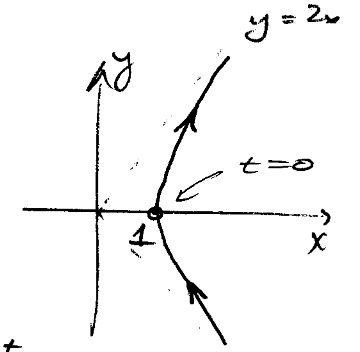
(b) $x = 3\sin t$, $y = \cos t \Rightarrow \left(\frac{x}{3}\right)^2 + y^2 = 1$

ellipse



(c) $x = \cosh t$, $y = 2\sinh t \Rightarrow x^2 - \left(\frac{y}{2}\right)^2 = 1$

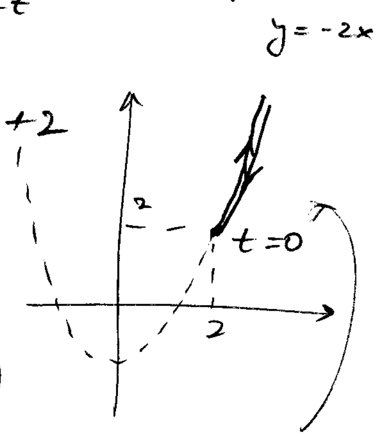
hyperbola $(x > 0)$



(d) $x^2 = (e^t + e^{-t})^2 = e^{2t} + e^{-2t} + 2e^t \cdot e^{-t}$

$= (e^{2t} + e^{-2t}) + 2 = y + 2$

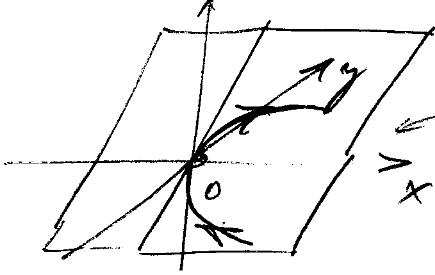
$y = x^2 - 2$ - parabola



Curve traveled twice, for $t > 0$, and for $t < 0$.

(e) $x = t^2$, $y = t$, $z = t$ $\Rightarrow$ $y = z$ (plane in $\mathbb{R}^3$)

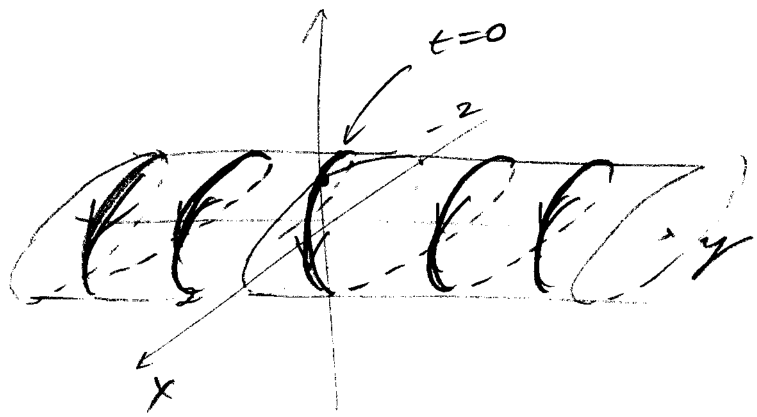
$x = y^2 = z^2$ (parabolas in the xy, xz planes)



the plane bisects the 1st quadrant on the yz-plane.

(f)
$$\begin{cases} x = 2\sin t \\ y = t \\ z = \cos t \end{cases} \Rightarrow \left(\frac{x}{2}\right)^2 + z^2 = 1$$

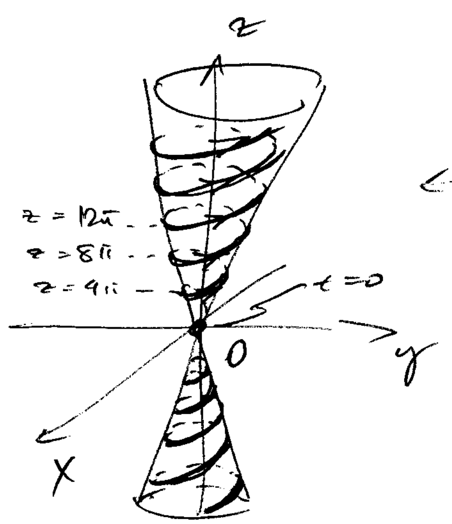
(elliptic cylinder)



(The curve coils up on the cylinder $\frac{x^2}{4} + z^2 = 1$)

2.
$$\left.\begin{aligned} x &= t\cos t \\ y &= t\sin t \\ z &= 2t \end{aligned}\right\} \Rightarrow \quad x^2 + y^2 = t^2(\cos^2 + \sin^2 t)$$
$$= t^2 = \frac{z^2}{4}$$

$x^2 + y^2 = \frac{z^2}{4}$ – circular cone



3.
$$\begin{cases} x = \sin t \\ y = \cos t \\ z = t \end{cases}$$

$x^2 + y^2 + z^2 = 5$

$\sin^2 t + \cos^2 t + t^2 = 5$

$1 + t^2 = 5$

$t^2 = 4$

$t = \pm 2$

$t = -2 \Rightarrow \quad (-\sin 2, \cos 2, -2)$

$t = 2 \Rightarrow \quad (\sin 2, \cos 2, 2).$

4.
$x = t^2 \quad = 1$
$y = 1 - 3t \quad = 4 \quad \Rightarrow t = -1$ ✓
$z = 1 + t^3 \quad = 0$

$= 9$
$= -8 \quad \Rightarrow \quad t = 3$ ✓
$= 28$

$t^2 = 4$ ✓
$1 - 3t = 7 \Rightarrow 3t = -6 \Rightarrow t = -2$ ✓
$1 + t^3 = -6$ ✗

System of equations has no solution.

5 (a) $x^2 + y^2 = 4 \Rightarrow \begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}$ is a param.

$z = xy \rightarrow z = 4 \sin t \cos t = 2 \sin(2t)$

$\vec{r}(t) = 2(\cos t, \sin t, \sin(2t)).$

(b) $x^2+z^2=1 \Rightarrow \begin{array}{l} x=\cos t \\ z=\sin t \end{array}$

$$y^2=4-4z^2-x^2 \quad , y\geq 0$$

$$\Rightarrow y=\sqrt{4-4z^2-x^2}$$

$$=\sqrt{4-4\sin^2 t-\cos^2 t}=\sqrt{3-3\sin^2 t}$$

$$=\sqrt{3\cos^2 t}=\sqrt{3}|\cos t|,$$

$$\vec{r}(t)=(\cos t, \sqrt{3}|\cos t|, \sin t).$$

(c) $z=\sqrt{x^2+y^2} \Rightarrow z^2=x^2+y^2$

$$z=\frac{1-y}{2} \Rightarrow z^2=\left(\frac{1-y}{2}\right)^2=\frac{1-2y+y^2}{4}$$

$$1-2y+y^2=4(x^2+y^2)$$

$$4x^2+3y^2+2y=1$$

$$4x^2+3\left(y^2+\frac{2}{3}y+\frac{1}{9}\right)-\frac{1}{3}=1$$

$$4x^2+3\left(y+\frac{1}{3}\right)^2=\frac{4}{3}$$

$$3x^2+\frac{9}{4}\left(y+\frac{1}{3}\right)^2=1$$

(ellipse)

$$x=\sqrt{3}\cos t$$

$$y+\frac{1}{3}=\frac{2}{3}\sin t \Rightarrow y=-\frac{1}{3}+\frac{2}{3}\sin t$$

$$z=\frac{1}{2}-\frac{1}{2}y=\frac{1}{2}+\frac{1}{6}-\frac{1}{3}\sin t$$

$$=\frac{1}{3}(1-\sin t)$$

$$\vec{r}(t)=\left(\sqrt{3}\cos t, -\frac{1}{3}+\frac{2}{3}\sin t, \frac{1}{3}(1-\sin t)\right).$$

(d) $z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$

$z = 1+2y \Rightarrow z^2 = (1+2y)^2 = 1+4y+4y^2$

$1+4y+4y^2 = x^2+y^2$

$1 = x^2-3y^2-4y = x^2-3\left(y^2+\frac{4}{3}y+\frac{4}{9}\right)+\frac{4}{3}$

$x^2-3\left(y+\frac{2}{3}\right)^2 = -\frac{1}{3}$

$9\left(y+\frac{2}{3}\right)^2-3x^2 = 1$

$$\begin{cases} 3\left(y+\frac{2}{3}\right) = \cosh t \\ \sqrt{3}\,x = \sinh t \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{3}}\sinh t \\ y = -\frac{2}{3}+\frac{1}{3}\cosh t \\ z = 1+2y \\ \quad = \frac{7}{3}+\frac{2}{3}\cosh t. \end{cases}$$

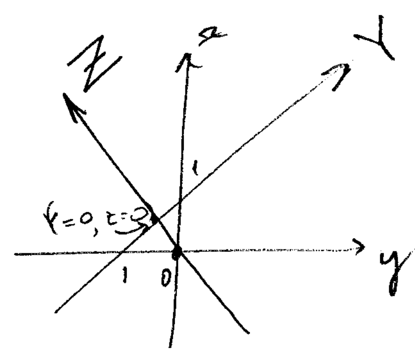
OR
$$\begin{cases} x = \frac{1}{\sqrt{3}}\sinh t \\ y = -\frac{2}{3}-\frac{1}{3}\cosh t \\ z = \frac{7}{3}-\frac{2}{3}\cosh t \end{cases}$$
(two branches of a hyperbola)

6 (a) $X = t$

$Y = \frac{1}{\sqrt{2}}\left(\frac{t^2-1}{2}+\frac{t^2+1}{2}\right) = \frac{t^2}{\sqrt{2}}$

$Z = \frac{1}{\sqrt{2}}\left(-\frac{t^2-1}{2}+\frac{t^2+1}{2}-1\right) = 0$

$\Rightarrow Z = 0, \quad Y = \frac{X^2}{\sqrt{2}}.$



8.

$x = t = 1+2t \Rightarrow t = -1$

$y = t^2 = 1+6t \quad \times$

$z = t^3 = 1+14t \quad \times$

system of equations has no solution $\Rightarrow$ particles do not collide.

$$\left.\begin{aligned} x = t &= 1+2s \\ y = t^2 &= 1+6s \\ z = t^3 &= 1+14s \end{aligned}\right\} \Rightarrow \begin{aligned} 2s &= t-1 \\ 2s &= \frac{t^2-1}{3} \\ 2s &= \frac{t^3-1}{7} \end{aligned}$$

$$t-1 = \frac{t^2-1}{3} = \frac{t^3-1}{7}$$

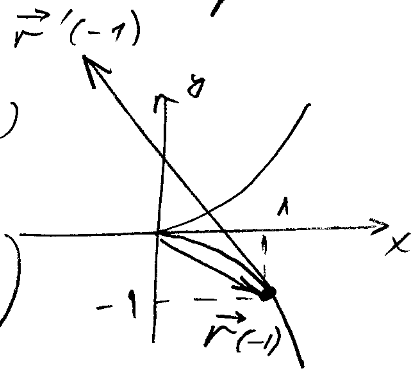
has 2 solutions $t=2 \Rightarrow s=\frac{1}{2}$

$t=1 \Rightarrow s=0$

$\Rightarrow$ their paths intersect at $(2,4,8)$ and $(1,1,1)$.

9. (a)

$\left(x = y^{2/3}\right)$

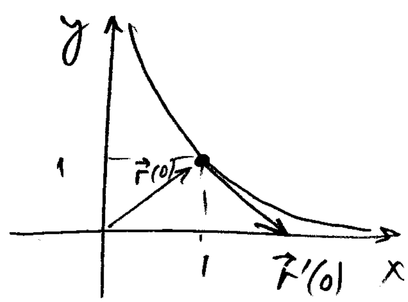


$\vec{r}'(t) = (2t, 3t^2)$

$r'(-1) = (-2, 3)$

(b)

$\left(y = \frac{1}{x},\ x>0\right)$

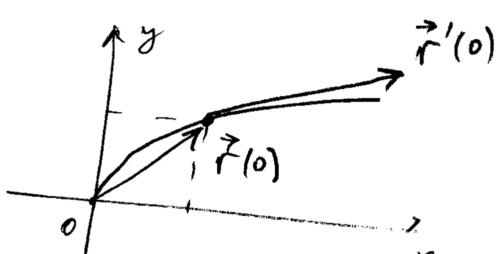


$\vec{r}'(t) = (e^t, -e^{-t})$

$\vec{r}'(0) = (1, -1)$.

(c)

$\left(x = y^2,\ x>0\right)$

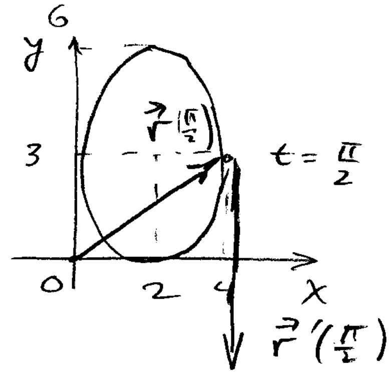


$\vec{r}'(t) = (2e^{2t}, e^t)$

$\vec{r}'(0) = (2, 1)$.

(d)

$$\frac{(x-2)^2}{2^2}+\frac{(y-3)^2}{3^2}=1$$



$$\vec{r}'(t)=(2\cos t,\ -3\sin t)$$
$$\vec{r}'(\tfrac{\pi}{2})=(0,\ -3)$$

10. (a) $\vec{r}'(t)=(\cos t,\ -2\sin t,\ 1)$

$\vec{r}'(\frac{\pi}{2})=(0,\ -2,\ 1)$

$\vec{T}(\frac{\pi}{2})=(0,\ -\frac{2}{\sqrt5},\ \frac{1}{\sqrt5})$.

$\vec{r}(\frac{\pi}{2})=(1,\ 0,\ \frac{\pi}{2})$

Tangent line:

$$\begin{cases} x=1 \\ y=-2t \\ z=\frac{\pi}{2}+t \end{cases}$$

(b) $\vec{r}'(t)=(\frac{1}{\sqrt t},\ 3t^2-1,\ 3t^2+1)$

$\vec{r}'(1)=(1,\ 2,\ 4)$

$\vec{T}(1)=(\frac{1}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{4}{\sqrt{21}})$

$\vec{r}(1)=(3,\ 0,\ 2)$

Tangent line:

$$\begin{cases} x=3+t \\ y=2t \\ z=2+4t \end{cases}$$

(c) $\vec{r}'(t)=\left(\frac{t}{\sqrt{t^2+3}},\ \frac{2t}{t^2+3},\ 1\right)$

$\vec{r}'(1)=(\frac12,\ \frac12,\ 1)$

$\vec{T}(1)=\frac{(1,1,2)}{\sqrt6}=(\frac{1}{\sqrt6},\ \frac{1}{\sqrt6},\ \frac{2}{\sqrt6})$

$\vec{r}(1)=(2,\ \ln 4,\ 1)$

Tangent line:

$$\begin{cases} x=2+\frac12 t \\ y=\ln 4+\frac12 t \\ z=1+t \end{cases}$$

(d) $\vec{r}\,'(t) = \left(-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t,\ -e^{-t}\right)$

$\vec{r}\,'(0) = (-1, 1, -1)$

$\vec{T}(0) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

$\vec{r}(0) = (1, 0, 1)$

Tangent line:

$$\begin{cases} x = 1 - t \\ y = t \\ z = 1 - t \end{cases}$$

11. $x = t \quad = 3 - s$

$y = 1 - t \quad = s - 2 \quad \Rightarrow s = 3 - t$

$z = 3 + t^2 \quad = s^2$

$$3 + \cancel{t^2} = (3-t)^2 = 9 - 6t + \cancel{t^2}$$

$$6t = 6 \Rightarrow t = 1 \quad \Rightarrow s = 2$$

Point of intersection: $(1, 0, 4)$

Tangent vectors:

$\vec{r}_1\,'(t) = (1, -1, 2t) \Rightarrow \vec{r}_1\,'(1) = (1, -1, 2)$

$\vec{r}_2\,'(t) = (-1, 1, 2s) \Rightarrow \vec{r}_2\,'(2) = (-1, 1, 4)$

$$\cos\vartheta = \frac{(1,-1,2)\cdot(-1,1,4)}{\sqrt{6}\,\sqrt{18}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\vartheta = \arccos\frac{1}{\sqrt{3}} \approx 55^\circ \quad \text{or} \quad 0.9553 \text{ rad.}$$

11

13. (a) $\vec{r}\,'(t) = \left(1, \frac{2t}{\sqrt{2}}, t^2\right)$

$|\vec{r}\,'(t)| = \sqrt{1+2t^2+t^4} = 1+t^2$

$\ell(C) = \int_0^2 1+t^2 = 2 + \left[\frac{t^3}{3}\right]_0^2 = 2 + \frac{8}{3} = \frac{14}{3}.$

(b) $\vec{r}\,'(t) = \left(1, 3t, \frac{9}{2}t^2\right)$

$|\vec{r}\,'(t)| = \sqrt{1+9t^2+\frac{81}{4}t^4} = 1+\frac{9}{2}t^2$

$\ell(C) = \int_0^2 \left(1+\frac{9}{2}t^2\right)dt = 2 + \frac{9}{2}\,\frac{8}{3}$

$= 2+12 = 14$

(c) $\vec{r}\,'(t) = (1, \sec t, \tan t)$

$|r'(t)| = \sqrt{1+\frac{1}{\cos^2 t}+\frac{\sin^2 t}{\cos^2 t}} = \frac{\sqrt{2}}{\cos t} = \sqrt{2}\cdot\sec t$

$\ell(C) = \int_0^{\pi/4} \sqrt{2}\sec t\, dt = \sqrt{2}\left[\ln(\sec t + \tan t)\right]_0^{\pi/4}$

$= \sqrt{2}\left(\ln(1+\sqrt{2}) - \ln 1\right) = \sqrt{2}\ln(1+\sqrt{2})$

(e) $\vec{r}\,'(t) = \left(\sqrt{2}, e^t, -e^{-t}\right)$

$|r'(t)| = \sqrt{2+e^{2t}+e^{-2t}} = e^t+e^{-t}$

$\ell(C) = \int_0^1 e^t+e^{-t}\, dt = \left[e^t\right]_0^1 - \left[e^{-t}\right]_0^1$

$= (e-1)-(e^{-1}-1) = e-\frac{1}{e}.$