Homework for Sections 13.1, 13.2 + arc length in 13.3

Due on Thursday, October 2, 2014, 3:30pm.

1. Sketch the curve with the given vector equation. Indicate the point on the curve that corresponds to $t = 0$, and show with an arrow the direction in which t increases.

   (a) $\vec{r}(t) = (t^3, t^2)$

   (b) $\vec{r}(t) = (3\sin t, \cos t)$

   (c) $\vec{r}(t) = (\cosh t, 2\sinh t)$

   (d) $\vec{r}(t) = (e^t + e^{-t}, e^{2t} + e^{-2t})$; *Hint: Compute the square of* $e^t + e^{-t}$

   (e) $\vec{r}(t) = (t^2, t, t)$; *Hint: The curve lies in the plane* $y = z$

   (f) $\vec{r}(t) = (2\sin t, t, \cos t)$ *Hint: The curve lies on the cylinder* $\frac{x^2}{4} + z^2 = 1$.

2. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = 2t$ lies on the cone $x^2 + y^2 - \frac{z^2}{4} = 0$, and use this fact to help sketch the curve.

3. At what points does the helix with parametrization $\vec{r}(t) = (\sin t, \cos t, t)$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

4. Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

5. Find a parametrization for the curve of intersection of the two surfaces:

   (a) The cylinder $x^2 + y^2 = 4$ and the paraboloid $z = xy$

   (b) The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geq 0$ and the cylinder $x^2 + z^2 = 1$

   (c) The half-cone $z = \sqrt{x^2 + y^2}$ and the plane $2z = 1 - y$

   (d) The half-cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + 2y$.

6. (a) As discussed in class, the curve of intersection of the half-cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$ is a parabola, with parametric equations

$$x = t, \quad y = \tfrac{t^2 - 1}{2}, \quad z = \tfrac{t^2 + 1}{2}.$$

Introduce new coodinates according to the following:

$$
\begin{aligned}
X &= x \\
Y &= \tfrac{y}{\sqrt{2}} + \tfrac{z}{\sqrt{2}} \\
Z &= -\tfrac{y}{\sqrt{2}} + \tfrac{z}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}
\end{aligned}
$$

(Remark: the new coordinates correspond to rotation through $45°$ *about the* x *axis and a parallel-translation by* $(0, -\frac{1}{2}, \frac{1}{2})$*; the plane* $z = 1 + y$ *is transformed into* $Z = 0$*, the perpendicular plane* $y + z = 0$ *is transformed into* $Y = 0$*).* Show that in the new coodrinates the equations of the parabola take the form

$$Z = 0, \quad Y = \tfrac{1}{\sqrt{2}} X^2.$$

(b) Consider the curve described in problem 5 (d). Show that by introducing new coordinates as follows:

$$
\begin{aligned}
X &= x \\
Y &= \tfrac{y}{\sqrt{5}} + \tfrac{2z}{\sqrt{5}} + \tfrac{4}{3\sqrt{5}} \\
Z &= -\tfrac{2y}{\sqrt{5}} + \tfrac{z}{\sqrt{5}} - \tfrac{1}{\sqrt{5}}
\end{aligned}
$$

the equations of the curve of intersection can be reduced to the form

$$Z = 0, \quad \frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1 \quad (Y > 0)$$

(determine the values of a and b).

7. (bonus) Consider the curve described in problem 5 (c). Determine the parameters $\cos\theta$, $\sin\theta$, Y_0, Z_0 in the transformation

$$
\begin{aligned}
X &= x \\
Y &= y\cos\theta + z\sin\theta + Y_0 \\
Z &= -y\sin\theta + z\cos\theta + Z_0
\end{aligned}
$$

so that the equations of the curve are reduced to the following:

$$Z = 0, \quad 3X^2 + \tfrac{9}{5}Y^2 = 1.$$

8. Two particles travel along the space curves

$$\vec{r_1}(t) = (t, t^2, t^3), \quad \vec{r_2}(t) = (1 + 2t, 1 + 6t, 1 + 14t).$$

(Here t is the common parameter which is interpteted as time.) Do the particles collide? Do their paths intersect?

9. (i) Sketch the plane curve with the given parametrization. (ii) Find $\vec{r}\,'(t)$. (iii) Sketch the position vector $\vec{r}(t_0)$ and the tangent vector $\vec{r}\,'(t_0)$ for the given value of t_0.

   (a) $\vec{r}(t) = (t^2, t^3)$, $t_0 = -1$

   (b) $\vec{r}(t) = (e^t, e^{-t})$, $t_0 = 0$

   (c) $\vec{r}(t) = (e^{2t}, e^t)$, $t_0 = 0$

   (d) $\vec{r}(t) = (2 + 2\sin t, 3 + 3\cos t)$, $t_0 = \pi/2$

10. Given a parametrization of a space curve: (i) find the derivative $\vec{r}\,'(t)$; (ii) find the tangent vector $\vec{r}\,'(t_0)$ for the given value of t_0; (iii) find the unit tangent vector $\vec{T}(t_0)$; (iv) find parametric equations for the tangent line to the curve at the point corresponding to the value $t = t_0$.

    (a) $\vec{r}(t) = (\sin t,\ 2\cos t,\ t)$, $t_0 = \pi/2$

    (b) $\vec{r}(t) = (1 + 2\sqrt{t},\ t^3 - t,\ t^3 + t)$, $t_0 = 1$

    (c) $\vec{r}(t) = (\sqrt{t^2+3},\ \ln(t^2+3),\ t)$, $t_0 = 1$

    (d) $\vec{r}(t) = (e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t})$, $t_0 = 0$.

11. At what point in $\mathbb{R}^3$ do the curves $\vec{r_1}(t) = (t, 1-t, 3+t^2)$ and $\vec{r_2}(s) = (3-s, s-2, s^2)$ intersect? Find the angle of their intersection, correct to the nearest degree.

12. (a) Given $f(t) = \vec{u}(t)\cdot\vec{v}(t)$, $\vec{u}(2) = (1, 2, -1)$, $\vec{u}\,'(2) = (3, 0, 4)$, and $\vec{v}\,'(t) = (t, t^2, t^3)$. Find $f'(2)$.

    (b) Suppose that $f(t) = \vec{u}(t)\times\vec{v}(t)$, and the rest of the data is the same as in part (a). Find $f'(2)$.

13. Find the length of the curve with the given parametrization:

    (a) $\vec{r}(t) = \left(t,\ t^2/\sqrt{2},\ \frac{1}{3}t^3\right)$, $\quad 0 \le t \le 2$

    (b) $\vec{r}(t) = \left(t,\ \frac{3}{2}t^2,\ \frac{3}{2}t^3\right)$, $\quad 0 \le t \le 2$

    (c) $\vec{r}(t) = \left(t,\ \ln(\sec t + \tan t),\ \ln\sec t\right)$, $\quad 0 \le t \le \pi/4$

    (d) $\vec{r}(t) = \left(t\cos t,\ t\sin t,\ t\right)$, $\quad 0 \le t \le \pi/2$

    (e) $\vec{r}(t) = \left(\sqrt{2}\,t, e^t, e^{-t}\right)$, $\quad 0 \le t \le 1$.