Wed, Nov 20, 2013 **MATH 250, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

This test includes 7 questions (total of 42 points), on 7 pages. The duration of the test is 75 minutes.

**Your scores:** (do not enter answers here)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | total |
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**Important:** The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) Verify that the function $f(x,y) = \ln\sqrt{x^2+y^2}$ is a solution of the differential equation $f_{xx} + f_{yy} = 0$.

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2)$$

$$f_x = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$f_y = \frac{1}{2}\,\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

$$f_{xx} = \frac{1}{x^2+y^2} - \frac{x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{(by symmetry.)}$$

$$f_{xx}+f_{yy} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

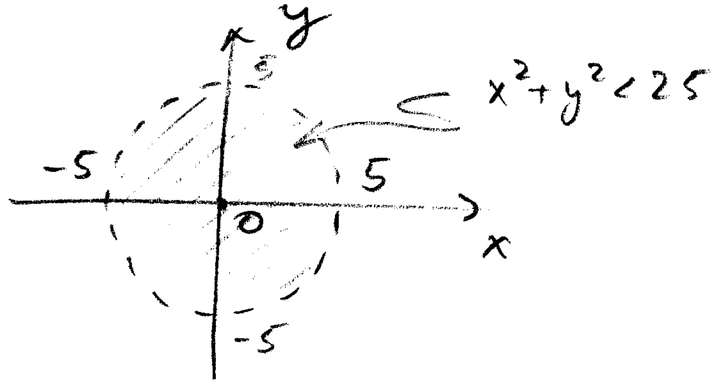
$$((x,y) \neq (0,0)).$$

2. (6 points) Given the function

$$g(x,y) = \frac{1}{\sqrt{25 - x^2 - y^2}}.$$

(a) Find the domain $D$ of the function $g(x,y)$ and sketch it.

$D = \{(x,y) : 25 - x^2 - y^2 > 0\} = \{(x,y) : x^2 + y^2 < 25\}$



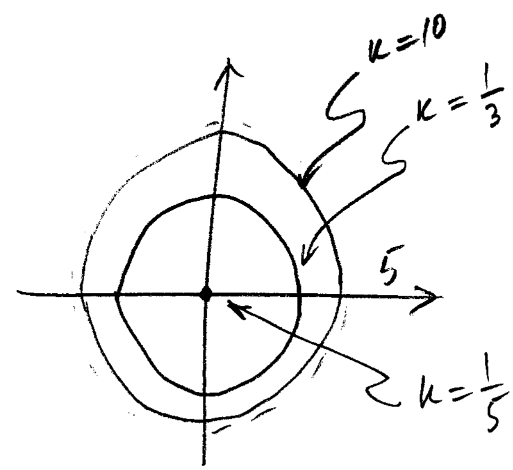

Points inside the circle of radius 5 about the origin (the boundary is not included)

(b) Sketch level curves $g(x,y) = k$ for $k = \frac{1}{5}, \frac{1}{3}$, and $k = 10$.

$k = \frac{1}{5}$: $\sqrt{25 - x^2 - y^2} = 5$
$\iff (x,y) = (0,0)$

$k = \frac{1}{3}$: $\sqrt{25 - x^2 - y^2} = 3$
$25 - x^2 - y^2 = 9$
$x^2 + y^2 = 16$
(circle of radius 4)

$k = 10$: $\sqrt{25 - x^2 - y^2} = \frac{1}{10}$
$x^2 + y^2 = 24.99$ (circle of radius just below 5)




(c) Based on the level curves, what can be said about the global minimum of $g(x,y)$ on $D$? The global maximum?

Global minimum $z = \frac{1}{5}$ at $(0,0)$

No global maximum:
$g(x,y)$ keeps increasing as $(x,y)$ approaches the circle $x^2 + y^2 = 25$.

*Continued...*

3. (6 points) Find the limits, or show that they do not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$

$$f(x,y) = \frac{4xy}{x^2+y^2}$$

$x=0$ or $y=0$ $\Rightarrow f(x,y) = 0$

$$x=y \Rightarrow f(x,y) = \frac{4x^2}{2x^2} = 2 \quad (x \neq 0)$$

Different values are approached from different directions as $(x,y) \to (0,0)$

$\Rightarrow$ the limit Does Not Exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-2x^2y^2}{x^2+y^2}$

$$g(x,y) = \frac{x^2+y^2-2x^2y^2}{x^2+y^2} = 1 - \frac{2x^2y^2}{x^2+y^2} = 1 - \underbrace{2xy}_{\to 0}\underbrace{\left(\frac{xy}{x^2+y^2}\right)}_{-\frac{1}{2} \le \ \cdot \ \le \frac{1}{2}}$$

Use squeeze theorem to show $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y) = 1$:

$$1-|x||y| = 1-2|x||y|\cdot\frac{1}{2} \le g(x,y) = 1 - 2xy\frac{xy}{x^2+y^2} \le 1+2|x||y|\cdot\frac{1}{2} = 1+|x||y|$$

$$1-\underbrace{|x||y|}_{\to 0} \le g(x,y) \le 1+\underbrace{|x||y|}_{\to 0}$$

*Continued...*

4. (6 points) (a) Find the unit vector $\vec{u}$ which determines the direction of fastest increase of the function $f$, locally at the given point $P$:

$$f(x,y) = (x^2+y)\,e^{y/2}, \quad P(1,0).$$

$$\nabla f = \left(2x e^{y/2},\ e^{y/2} + \tfrac{1}{2}(x^2+y)e^{y/2}\right)$$

$$= e^{y/2}\left(2x,\ 1+\tfrac{1}{2}x^2+\tfrac{1}{2}y\right)$$

$$\nabla f(1,0) = \left(2, \tfrac{3}{2}\right)$$

$$|\nabla f(1,0)| = \sqrt{2^2+\left(\tfrac{3}{2}\right)^2} = \sqrt{\frac{16+9}{4}} = \frac{5}{2}$$

$$\vec{u} = \frac{\nabla f(1,0)}{|\nabla f(1,0)|} = \left(\tfrac{4}{5}, \tfrac{3}{5}\right)$$

(b) Find the maximum rate of change for the problem in part (a).

$$|\nabla f(1,0)| = \frac{5}{2} = 2.5.$$

5. (6 points) Show that the point $(x_0, y_0) = (1, +1)$ is a critical point of the function

$$f(x,y) = x^3 - y^3 + 3x^2y - 9x$$

and determine its type (local maximum/minimum, or saddle point).

$$\nabla f = \left(3x^2 + 6xy - 9,\ -3y^2 + 3x^2\right)$$

$\nabla f(1,1) = (0,0) \Rightarrow (1,1)$ is a crit. pt.

$$D^2 f = \begin{pmatrix} 6x+6y & 6x \\ 6x & -6y \end{pmatrix}$$

$$D^2 f(1,1) = \begin{pmatrix} 12 & 6 \\ 6 & -6 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\Delta = AC - B^2 = 12 \cdot (-6) - 6^2$$
$$= -108 < 0$$

$\Rightarrow (1,1)$ is a saddle point.

6. (6 points) Use Lagrange's method to find the maximum and minimum of $f(x, y)$, subject to the given constraint:

$$f(x, y) = y^2 - x^2, \quad \text{constraint:} \quad g(x, y) = 4x^2 + y^2 = 1.$$

$\nabla f = (-2x, +2y)$

$\nabla g = (8x, 2y)$

$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} -2x = 8\lambda x \Rightarrow x = 0 \text{ or } \lambda = -\frac{1}{4} \\ 2y = 2\lambda y \Rightarrow y = 0 \text{ or } \lambda = 1 \end{cases}$

$+$ constraint $\quad 4x^2 + y^2 = 1$

$x = 0 \Rightarrow y = \pm 1$

$y = 0 \Rightarrow x = \pm \frac{1}{2}$ ;

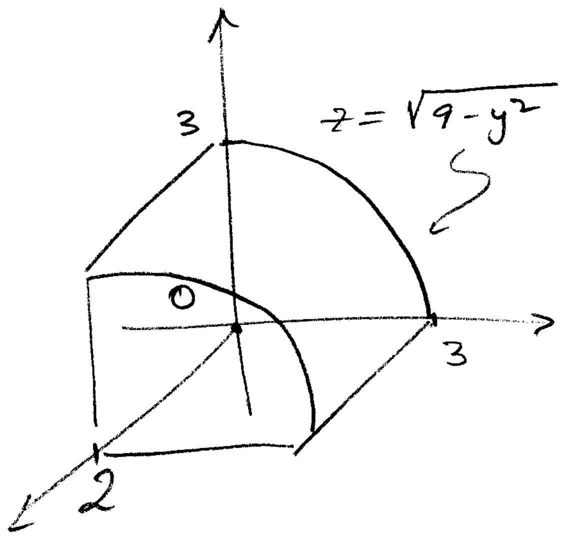
$x \neq 0$ or $y \neq 0 \Rightarrow$ the system has no solution.

Minimum at $(\pm\frac{1}{2}, 0) \Rightarrow f(\pm\frac{1}{2}, 0) = -\frac{1}{4}$

Maximum at $(0, \pm 1) \Rightarrow f(0, \pm 1) = 1.$

7. (6 points) (a) Sketch the three-dimensional region whose volume is represented by the integral

$$\iint_R \sqrt{9-y^2}\,dx\,dy, \qquad R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 3\}.$$




Under the half-cylinder

$z = \sqrt{9-y^2}$

above the rectangle

$[0,2] \times [0,3]$

(b) Evaluate the integral in (a) using either elementary geometry or calculus.

Geometry: $V = Ah = \frac{1}{4}\pi \cdot r^2 h = \frac{9\pi}{2}$

$(h=2)\ (r=3)$

Calculus: $\iint_R \sqrt{9-y^2}\,dx\,dy = \int_0^3 \int_0^2 \sqrt{9-y^2}\,dx\,dy$

$= \int_0^3 \sqrt{9-y^2} \underbrace{\int_0^2 dx}_{=2}\, dy$

$= 2\int_0^3 \sqrt{9-y^2}\,dy = 2 \cdot \int_0^{\pi/2} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\,d\theta$

$\left( \begin{matrix} y = 3\sin\theta \\ dy = 3\cos\theta\,d\theta \end{matrix} \right)$

int. = 0.

THE END.

$= 18 \cdot \int_0^{\pi/2} \cos^2\theta\,d\theta = 18 \cdot \frac{1}{2} \int_0^{\pi/2} (1 + \cos(2\theta))\,d\theta = \frac{9\pi}{2}$