Wed, Oct 9, 2013 **MATH 250, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

This test includes 9 questions (total of 58 points), on 9 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) (a) Find an equation of the sphere that passes through the point $P(1,1,1)$ and has center at $C(1,2,-1)$.

$|CP| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$ – radius of the sphere.

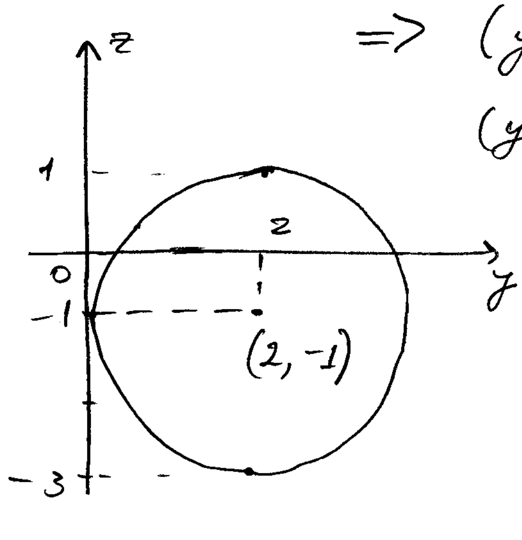
$(x-1)^2+(y-2)^2+(z+1)^2 = 5.$

(b) Find an equation of the curve in which the sphere intersects the yz-plane. Sketch the curve in the plane.

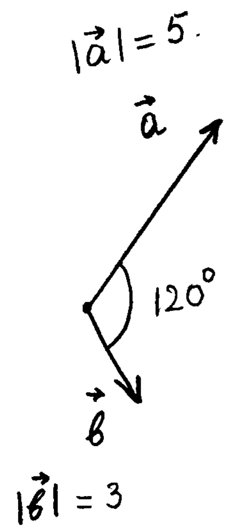
yz-plane: $\{x=0\}$

$\Rightarrow (y-2)^2+(z+1)^2+1=5$

$(y-2)^2+(z+1)^2=4$ – circle of radius 2 about $(2,-1)$



2. (4 points) If $\vec{a}$ and $\vec{b}$ are vectors shown in the figure, find $\vec{a}\cdot\vec{b}$ and $|\vec{a}\times\vec{b}|$. Is $\vec{a}\times\vec{b}$ directed into the page or out of it?



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 120^\circ = 15\cdot\left(-\frac{1}{2}\right) = -\frac{15}{2}.$$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin 120^\circ = \frac{15\sqrt{3}}{2}$$

$\vec{a} \curvearrowright \vec{b}$ — clockwise $\Rightarrow$

$\vec{a}\times\vec{b}$ is directed into the page.

3. (6 points) (a) Use properties of the dot product to show that for *any* vectors $\vec{u}$ and $\vec{v}$

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2.$$

distributive

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot(\vec{u}-\vec{v}) + \vec{v}\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \underbrace{\cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{v}\cdot\vec{u}}}_{\text{symmetry}} - \vec{v}\cdot\vec{v}$$

distributive

$$= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} = |\vec{u}|^2 - |\vec{v}|^2$$

(b) If $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, show that $\vec{u}$ and $\vec{v}$ must have the same length.

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$$

$$|\vec{u}|^2 - |\vec{v}|^2 = 0$$

$$\Rightarrow |\vec{u}|^2 = |\vec{v}|^2 \Rightarrow |\vec{u}| = |\vec{v}|$$ — they have the same length

Continued...

4. (6 points) Determine whether the line with symmetric equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

is parallel to the plane $2x + 3y + 4z + 9 = 0$. If not parallel, find the point of their intersection.

$\vec{n} = (2,3,4)$ – normal vector.

$\vec{v} = (2,3,4)$ – direction vector of the plane.

$\vec{n} \parallel \vec{v} \Rightarrow$ the line is perpendicular to the plane.

Point of intersection:

Vector form for the line:

$$(x,y,z) = (1,2,3) + t(2,3,4)$$

Substitute into $2x+3y+4z+9=0$

$$\Rightarrow 2(1+2t) + 3(2+3t) + 4(3+4t) + 9 = 0$$

$$2+6+12+9 + 4t+9t+16t = 0$$

$$29 + 29t = 0$$

$$\Rightarrow t = -1$$

$$\Rightarrow (x,y,z) = (1,2,3) - (2,3,4) = \underline{\underline{(-1,-1,-1)}}.$$

Continued...

5. (6 points) Find a vector function that represents the curve of intersection of the two surfaces:

The hyperboloid $x^2 + y^2 - z^2 = 1$, and the plane $x + y + z = 1$.

Hint: Start by setting $x = t$.

$x = t$

$y = 1 - x - z = 1 - t - z$

$z^2 = x^2 + y^2 - 1 = t^2 + (1 - t - z)^2 - 1$

$= t^2 - 1 + (1-t)^2 - 2z(1-t) + z^2$

$= t^2 - 1 + 1 + t^2 - 2t - 2(1-t)z + z^2$

$= 2t^2 - 2t - 2(1-t)z + z^2$

$\Rightarrow \quad 2t(t-1) - 2(1-t)z = 0$

$\Rightarrow \quad z = -t \quad$ or $\quad t = 1$

$$\Rightarrow \quad \begin{cases} x = t \\ y = 1 - t + t = 1 \\ z = -t \end{cases} \Rightarrow \quad \begin{cases} x = t \\ y = 1 \\ z = -t. \end{cases}$$

Continued...

6. (8 points) Consider the curve C given parametrically by the vector function

$$\vec{r}(t) = (\sqrt{2}\sin t, \sqrt{2}\cos t, 2\ln\cos t)$$

(a) Find the derivative $\vec{r}'(t)$.

$$\vec{r}'(t) = \left(\sqrt{2}\cos t, -\sqrt{2}\sin t, 2\frac{(-\sin t)}{\cos t}\right)$$
$$= (\sqrt{2}\cos t, -\sqrt{2}\sin t, -2\tan t).$$

(b) Find a vector equation for the line L tangent to the curve C at the point $P(1, 1, -\ln 2)$.

$\to t = \frac{\pi}{4}$ $\left(\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \ln\cos\frac{\pi}{4} = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2 \right)$

$$\vec{r}'\left(\frac{\pi}{4}\right) = (1, -1, -2)$$
$$\vec{r}\left(\frac{\pi}{4}\right) = (1, 1, -\ln 2)$$

Line: $\vec{r} = (1, 1, -\ln 2) + t(1, -1, -2)$.

(c) Find the point at which the line L intersects the xy-plane.

xy-plane: $\{z = 0\}$

$$z = -\ln 2 - 2t = 0 \Rightarrow t = -\frac{\ln 2}{2}$$

$$\Rightarrow x = 1 - \frac{\ln 2}{2}, \quad y = 1 + \frac{\ln 2}{2} \Rightarrow P\left(1 - \frac{\ln 2}{2}, 1 + \frac{\ln 2}{2}, 0\right).$$

7. (8 points) Consider the equation of a quadric surface:

$$4x^2 + y^2 - 4z^2 - 4y + 8 = 0.$$

(a) Reduce the equation to one of the standard forms.

$$4x^2 + (y^2 - 4y + 4) - 4 - 4z^2 + 8 = 0$$

$$4x^2 + (y-2)^2 - 4z^2 + 4 = 0$$

$$x^2 + \frac{(y-2)^2}{4} - z^2 = -1$$

(elliptic hyperboloid in two sheets.)

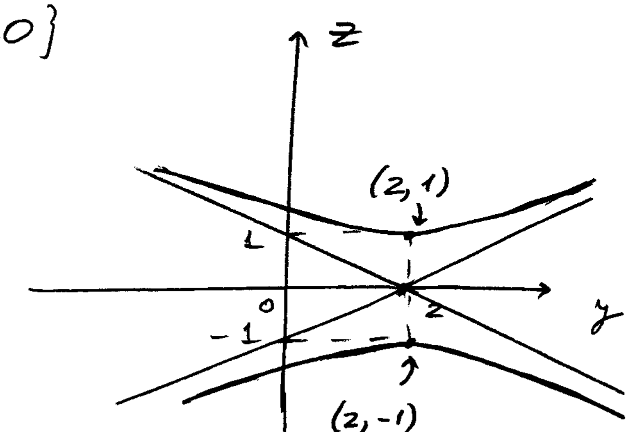
(b) Find an equation for the section of the surface by the yz-plane and sketch it.

yz-plane — $\{x=0\}$

$$\frac{(y-2)^2}{4} - z^2 = -1$$

$$z^2 - \frac{(y-2)^2}{4} = 1$$

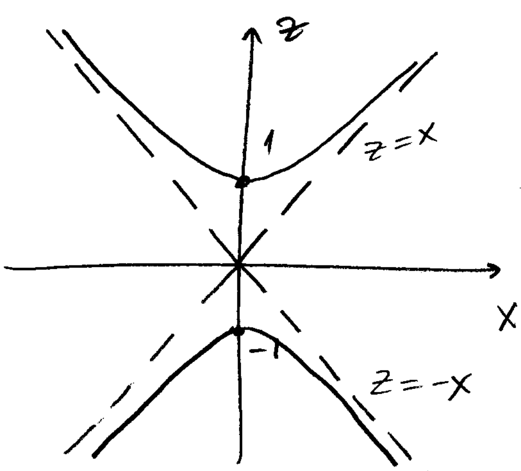
– hyperbola with asymptotes $z = \pm \frac{y-2}{2}$ (corresponds to $|z| > \frac{|y-2|}{2}$



(c) Find an equation for the section of the surface by the plane $y = 2$ and sketch it.

$\{y=2\} \Rightarrow x^2 - z^2 = -1 \iff z^2 - x^2 = 1$

hyperbola with asymptotes $z = \pm x$ (corresponds to $|z| > |x|$)



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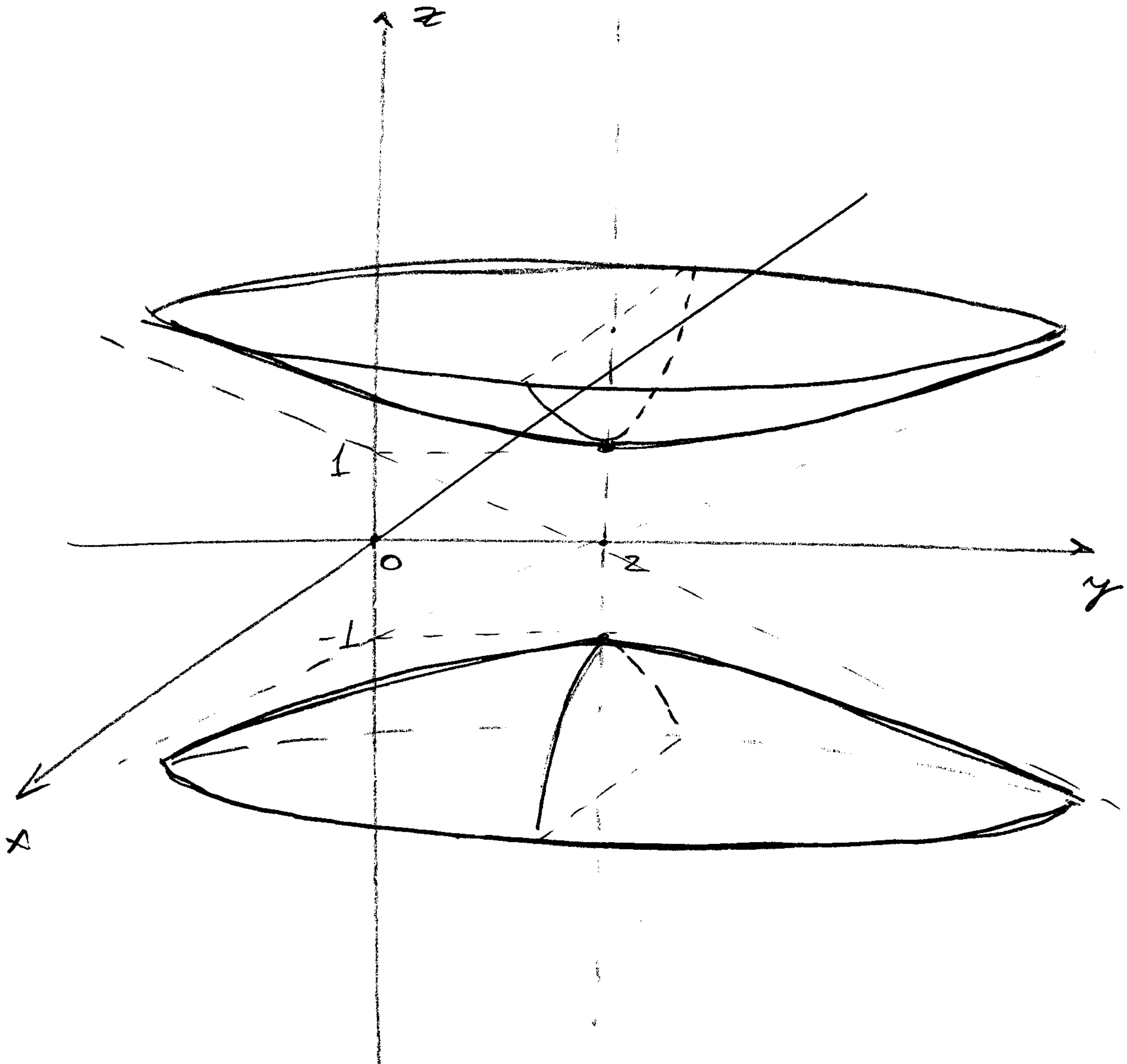
(d) Use the information obtained in parts (a), (b), (c) to make an accurate sketch of the surface. Show the sections by the planes from (b) and (c) on your graph. Classify the surface.



elliptic hyperboloid
in two sheets.

8. (6 points) Find the length of the curve:

$$\vec{r}(t) = \sqrt{2}\,t\,\vec{i} + e^t\,\vec{j} + e^{-t}\,\vec{k}, \quad 0 \le t \le \ln 2.$$

$$\vec{r}\,'(t) = \sqrt{2}\,\vec{i} + e^t\,\vec{j} - e^{-t}\,\vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{\sqrt{2}^{\,2} + e^{2t} + e^{-2t}}$$

$$= \sqrt{2 + e^{2t} + e^{-2t}}$$

$$= \sqrt{(e^t)^2 + 2e^te^{-t} + (e^{-t})^2}$$

$$= \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$\ell(C) = \int_0^{\ln 2} e^t + e^{-t}\,dt = \left[e^t - e^{-t}\right]_0^{\ln 2}$$

$$= (2 - \cancel{1}) - \left(\frac{1}{2} - \cancel{1}\right) = 2 - \frac{1}{2} = \frac{3}{2}.$$

9. (8 points) Find the vectors $\vec{T}(t)$, $\vec{N}(t)$ and the curvature $\kappa(t)$:

$$\vec{r}(t) = (t, 3\cos t, 3\sin t).$$

$$\vec{r}'(t) = (1, -3\sin t, 3\cos t)$$

$$|\vec{r}'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t}$$

$$= \sqrt{10}.$$

$$\vec{T}(t) = \frac{1}{\sqrt{10}}(1, -3\sin t, 3\cos t)$$

$$\vec{T}'(t) = \left(0, -\frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t\right)$$

$$\vec{\kappa}(t) = \frac{\vec{T}'(t)}{|\vec{r}'(t)|} = \left(0, -\frac{3}{10}\cos t, -\frac{3}{10}\sin t\right)$$

$$\kappa(t) = \frac{3}{10}\sqrt{\cos^2 t + \sin^2 t} = \frac{3}{10}.$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = (0, -\cos t, -\sin t)$$

$\vec{N}$ is parallel to the yz-plane