Solution of Navier-Stokes Equations – Part Three

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Mechanical Engineering 692

Computational Fluid Dynamics

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Outline

- Review SIMPLE method for integration of incompressible Navier-Stokes
  - Note choice of method is independent of choice of differencing scheme
- Discuss approaches for other methods
  - SIMPLEC
  - SIMPLER
  - PISO
- Discuss advantages or disadvantages of each method

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But First a Word on Project

- Proposals due Wednesday, March 17
- Most students will use Fluent at CSUN
- Can also use software at your job if available
- Objective is to exercise CFD program over range of options
  - Different grid sizes interesting, but difficult
  - Different turbulence models
  - Different algorithms

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More on Project

- Constant *versus* variable properties
- Can look at simple or more complicated flows involving multiple species or phases or chemical reaction
- Suggest tutorial projects in Fluent or Gambit as starting points
  - Gambit is used to create meshes that can be used in Fluent
  - Tutorials there give you background in generating CFD meshes

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What is SIMPLE?

- An approach for solving finite-volume equations for velocity components and pressure
  - Uses finite volume continuity and momentum equations
  - Uses correct terms, u, v, p, incorrect terms u*, v*, p*, and correction terms u', v', p'
    - u = u* + u', v = v* + v', p = p* + p'
  - Basic idea is to link continuity and momentum finite-volume equations to get equation for correction pressure

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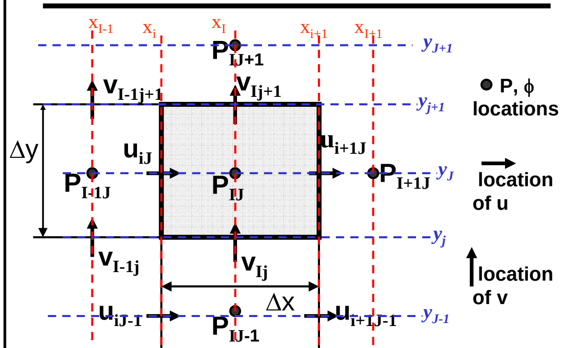
Review Staggered Grid



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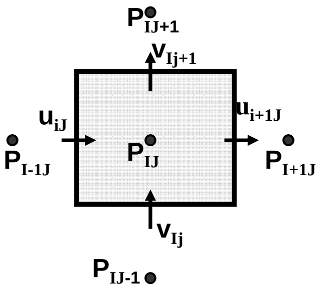
Review Continuity Equation



$$\frac{\rho_{I+1J}+\rho_{IJ}}{2}(uA)_{i+1J} - \frac{\rho_{IJ}+\rho_{I-1J}}{2}(uA)_{iJ} + \frac{\rho_{IJ+1}+\rho_{IJ}}{2}(vA)_{Ij+1} - \frac{\rho_{IJ}+\rho_{IJ-1}}{2}(vA)_{Ij} = 0$$

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Correction Pressure Equation

- Links correction pressure at central node to that at four neighbors

$$a_{I+1J}\,p'_{I+1J} + a_{I-1J}\,p'_{I-1J} + a_{IJ+1}\,p'_{IJ+1} + a_{IJ-1}\,p'_{IJ+1} - a_{IJ}\,p'_{IJ} = b'_{IJ}$$

$$a_{IJ} = a_{I+1J} + a_{I-1J} + a_{IJ+1} + a_{IJ-1}$$

- Source term is continuity error linking correction pressures to velocity

$$b_{IJ} = -\frac{\rho_{I+1J}+\rho_{IJ}}{2}A_{i+1J}u^*_{i+1J} + \frac{\rho_{IJ}+\rho_{I-1J}}{2}A_{iJ}u^*_{iJ} - \frac{\rho_{IJ+1}+\rho_{IJ}}{2}A_{Ij+1}v^*_{Ij+1} + \frac{\rho_{IJ}+\rho_{IJ-1}}{2}A_{Ij}v^*_{Ij}$$

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Correction Pressure Equation II

$$a_{I+1J}\,p'_{I+1J} + a_{I-1J}\,p'_{I-1J} + a_{IJ+1}\,p'_{IJ+1} + a_{IJ-1}\,p'_{IJ+1} - a_{IJ}\,p'_{IJ} = b'_{IJ}$$

- Coefficients linked to momentum d terms

$$d_{i+1J} = \frac{A_{i+1J}}{a_{i+1J}} \qquad a_{I+1J} = \frac{\rho_{I+1J}+\rho_{IJ}}{2}A_{i+1J}d_{i+1J}$$

$$d_{iJ} = \frac{A_{iJ}}{a_{iJ}} \qquad a_{I-1J} = \frac{\rho_{IJ}+\rho_{I-1J}}{2}A_{iJ}d_{iJ}$$

$$d_{Ij+1} = \frac{A_{Ij+1}}{a_{Ij+1}} \qquad a_{IJ+1} = \frac{\rho_{IJ+1}+\rho_{IJ}}{2}A_{Ij+1}d_{Ij+1}$$

$$d_{Ij} = \frac{A_{Ij}}{a_{Ij}} \qquad a_{IJ-1} = \frac{\rho_{IJ}+\rho_{IJ-1}}{2}A_{Ij}d_{Ij}$$

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Momentum Terms

$$a_{iJ}u_{iJ} = a_N u_{iJ+1} + a_S u_{iJ-1} + a_E u_{i+1J} + a_W u_{i-1J} - (p_{IJ} - p_{I-1J})A_{iJ} - b_{iJ}$$

$$u_{iJ} = \frac{\sum_{nb} a_{nb}u_{nb}}{a_{iJ}} + (p_{I-1J} - p_{IJ})\frac{A_{iJ}}{a_{iJ}} - \frac{b_{iJ}}{a_{iJ}}$$

$$a_{Ij}v_{Ij} = a_N v_{Ij+1} + a_S v_{iJ-1} + a_E v_{I+1j} + a_W v_{I-1j} - (p_{IJ} - p_{IJ-1})A_{Ij} - b_{Ij}$$

$$v_{Ij} = \frac{\sum_{nb} a_{nb}u_{nb}}{a_{Ij}} + (p_{IJ-1} - p_{IJ})\frac{A_{Ij}}{a_{Ij}} - \frac{b_{iJ}}{a_{Ij}}$$

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Review Momentum Coefficients

- a_K terms in momentum equations formed from F = ρu and D = Γ/δ terms
  - Can use any method such as central, upwind, hybrid, power-law, QUICK, to combine F and D
  - Coefficient values different for u and v and different across grid
  - Computation of coefficients depends on location of variables on staggered grid

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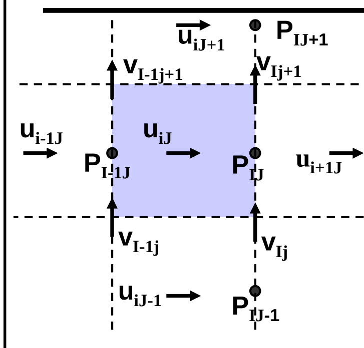
Control Volume for u (e face)



equations for e face at IJ

$$F_e = \frac{F_{i+1J}+F_{iJ}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{IJ}+\rho_{I+1J}}{2}\right)u_{i+1J} + \left(\frac{\rho_{I-1J}+\rho_{IJ}}{2}\right)u_{iJ}\right]$$

$$D_e = \frac{\Gamma_{IJ}}{x_{i+1}-x_i}$$

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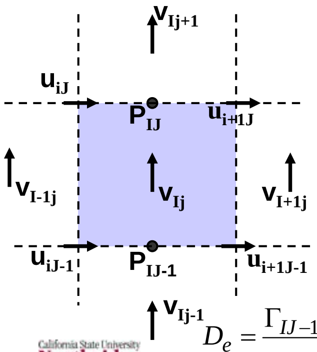
Control Volume for v (e face)



equations for e face at x_{i+1}

$$F_e = \frac{F_{I+1J} + F_{i+1J-1}}{2} = \frac{1}{2}\left[\left(\frac{\rho_{IJ} + \rho_{I+1J}}{2}\right)u_{i+1J} + \left(\frac{\rho_{IJ-1} + \rho_{I+1J-1}}{2}\right)u_{i+1J-1}\right]$$

$$D_e = \frac{\Gamma_{IJ-1} + \Gamma_{IJ} + \Gamma_{I+1J} + \Gamma_{I+1J-1}}{4\left(x_{I+1} - x_I\right)}$$

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Using Corrections

- Use of p = p* + p' can lead to divergence
  - Use underrelaxation: $p = p^* + \alpha_p p'$, where α_p is the underrelaxation factor ($0 < \alpha_p < 1$)
- Similar underrelaxation factors (between 0 and 1) for velocity correction

$$u_{iJ} = u_{iJ}^* + \alpha_u\left(p'_{I-1J} - p'_{IJ}\right)d_{iJ}$$

$$v_{Ij} = v_{Ij}^* + \alpha_v\left(p'_{IJ-1} - p'_{IJ}\right)d_{Ij}$$

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Putting it All Together

0. Start with initial guesses for p*, u*, v*
1. Start iterations
2. Compute coefficients in finite volume equations for momentum and pressure
3. Do an iteration on u* and v* equations
4. Get p' source term
5. Do an iteration on p' equation
6. Use p' to correct p*, u*, and v
7. Check convergence; if not converged return to step 1

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Other Variations

- SIMPLEC
  - SIMPLE Consistent
  - Small correction to p' equation
- SIMPLER
  - SIMPLE Revised
  - Has two correction equations
- PISO
  - Pressure Implicit with Splitting of Operators
  - With two correction equations, originally intended for transient problems

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SIMPLEC

- Almost the same as SIMPLE except for a small correction to the d terms
- Rationale: reduce the effect of neglecting the neighboring correction velocities in SIMPLE

$$d_{iJ} = \frac{A_{iJ}}{a_{iJ}} \Rightarrow d_{iJ} = \frac{A_{iJ}}{a_{iJ} - \sum a_{nb}}$$

$$d_{Ij} = \frac{A_{Ij}}{a_{Ij}} \Rightarrow d_{Ij} = \frac{A_{Ij}}{a_{Ij} - \sum a_{nb}}$$

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A Bit of History

- In 1967 Chorin proposed a method for solving the Navier-Stokes equation that solved a pressure equation and a velocity correction equation
- In SIMPLE, Patankar and Spaulding combined these for 3D boundary layers
- SIMPLER and PISO are similar to Chorin's approach
  - These methods require more work per step, but can converge more quickly

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A Bit of History II

- Chorin's basic idea: velocities for incorrect pressure have correct vorticity
  - Since $\omega = \nabla x\mathbf{v}$ and $\nabla x \nabla \lambda = 0$ for any λ, the difference between two velocity fields, **v** and **v*** (with same ω) must be $\nabla\lambda$.
  - To satisfy steady continuity, $\nabla\cdot\rho\mathbf{v} = 0$, with $\mathbf{v} = \mathbf{v}^* + \nabla\lambda, \nabla\cdot\rho(\mathbf{v}^* + \nabla\lambda) = 0$
  - The equation for the velocity corection potential is $\nabla\cdot\rho\nabla\lambda = -\nabla\cdot\rho\mathbf{v}^*$
  - The finite difference equation for λ is similar to the finite volume equation for p'

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SIMPLER

- Solves a finite-volume equation for pressure using current velocities and source terms
  - Equation is combination of momentum and continuity equations
  - Has same coefficients as in p' equation, but has different source term
- Solves momentum equation for u* and v* using pressures just found
- Solve p' equation only to correct u*, v*

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SIMPLER II

- Momentum equations for pressure

$$u_{iJ} = \frac{\sum_{nb} a_{nb}u_{nb} - b_{iJ}}{a_{iJ}} + (p_{I-1J} - p_{IJ})d_{iJ} = \hat{u}_{iJ} + (p_{I-1J} - p_{IJ})d_{iJ}$$

$$v_{Ij} = \frac{\sum_{nb} a_{nb}u_{nb} - b_{Ij}}{a_{Ij}} + (p_{IJ-1} - p_{IJ})d_{Ij} = \hat{v}_{Ij} + (p_{IJ-1} - p_{IJ})d_{Ij}$$

- Substitute velocities into continuity equation to get finite-volume equation for pressure

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SIMPLER Pressure Equation

- Links correction pressure at central node to that at four neighbors

a_K values same as in SIMPLE p' equation

$$a_{I+1J}\,p_{I+1J} + a_{I-1J}\,p_{I-1J} + a_{IJ+1}p_{IJ+1} + a_{IJ-1}p_{IJ+1} - a_{IJ}\,p_{IJ} = b_{IJ}$$

$$a_{IJ} = a_{I+1J} + a_{I-1J} + a_{IJ+1} + a_{IJ-1}$$

- Source term

$$b_{IJ} = -\frac{\rho_{I+1J} + \rho_{IJ}}{2}A_{i+1J}\hat{u}_{i+1J} + \frac{\rho_{IJ} + \rho_{I-1J}}{2}A_{iJ}\hat{u}_{iJ} - \frac{\rho_{IJ+1} + \rho_{IJ}}{2}A_{Ij+1}\hat{v}_{Ij+1} + \frac{\rho_{IJ} + \rho_{IJ-1}}{2}A_{Ij}\hat{v}_{Ij}$$

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SIMPLER Algorithm

0. Start with initial guesses for p*, u*, v*
1. Start iterations
2. Compute coefficients in finite volume equations for momentum and pressure
3. Compute u-hat and v-hat terms
4. Do an iteration on pressure
5. With p just found iterate momentum
6. Compute source terms for p'
7. Do an iteration on p' equation
8. Use p' to correct u* and v* only
9. Check convergence; if not converged return to step 1

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PISO Algorithm Introduction

- Originally designed for transient calculations with no iteration
- Iterative calculations seek a solution to a steady problem by solving simultaneous equations iteratively
- Transient calculations have an known initial state and compute the time behavior by taking small steps in Δt
- Transient calculations seek correct results at each time step

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PISO Overview

- Like SIMPLER, PISO starts with the basic approach of SIMPLE, but changes the correction procedure
- It makes two pressure corrections p’ and p’’
- In the second pressure correction the term $\Sigma a_{nb}u_{nb}$, which was set to 0 in SIMPLE, is approximated by velocities from the first pressure correction step

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PISO Details

- Start with a guessed pressure field p* and solve the momentum equations for u* and v* as in SIMPLE
- Correct the u* and v* fields by the same procedure used in SIMPLE
- Modify notation to accommodate further steps in simple
  - p** = p* + p’ replaces p = u* + p’
  - u** = u* + u’ replaces u = u* + u’
  - v** = v* + v’ replaces v = v* + v’

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More PISO Details

- Rewrite momentum equations to include neighbor velocities ignored in SIMPLE, but at old iteration step

$$u_{iJ}^{**} = \frac{\sum_{nb} a_{nb}u_{iJ}^{*} - b_{iJ}}{a_{iJ}} + \left(p_{I-1J}^{**} - p_{IJ}^{**}\right)d_{iJ}$$

$$v_{Ij}^{**} = \frac{\sum_{nb} a_{nb}v_{nb}^{*} - b_{Ij}}{a_{Ij}} + \left(p_{IJ-1}^{**} - p_{IJ}^{**}\right)d_{Ij}$$

- Get a second correction
  - p*** = p** + p”
  - u*** = u** + u” and v*** = v** + v”

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Still More PISO Details

- Momentum equations for u*** and v***

$$u_{iJ}^{***} = \frac{\sum_{nb} a_{nb}u_{iJ}^{**} - b_{iJ}}{a_{iJ}} + \left(p_{I-1J}^{***} - p_{IJ}^{***}\right)d_{iJ}$$

$$v_{Ij}^{***} = \frac{\sum_{nb} a_{nb}v_{nb}^{**} - b_{Ij}}{a_{Ij}} + \left(p_{IJ-1}^{***} - p_{IJ}^{***}\right)d_{Ij}$$

- Subtract u** and v** from u*** and v***

$$u_{iJ}^{***} - u_{iJ}^{**} = \frac{\sum_{nb} a_{nb}\left(u_{nb}^{**} - u_{nb}^{**}\right) - b_{iJ}}{a_{iJ}} + \left[\left(p_{I-1J}^{***} - p_{IJ}^{***}\right) - \left(p_{I-1J}^{**} - p_{IJ}^{**}\right)\right]d_{iJ}$$

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Last PISO Details Slide

- Resulting equations for u*** and v*** substituted into continuity to get finite-volume equation for p’’
- Solve p’’ equation and correct velocities and pressure
- Correct pressures and velocities
  - p*** = p** + p” = p* + p’ + p”
  - u*** = u** + $d_x\Delta$p”
  - v*** = v** + $d_y\Delta$p”

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PISO Algorithm

0. Start with initial guesses for p*, u*, v*
1. Start iterations
2. Compute coefficients in finite volume equations for momentum and pressure
3. Iterate momentum equations to update u* and v* using p* for pressure
4. Form and iterate equations for p’
5. Use p’ to get u** and v**
6. Form and iterate equations for p”
7. Use p” to get second corrections on pressure and velocities
8. Check convergence; if not converged return to step 1

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Comparison of Methods

- Each method solves the momentum equations for u and v plus some one or two other equations
  - Chorin's original 1967 approach solves for u, v, p, and velocity correction
  - SIMPLE, SIMPLEC solve for u, v, and p'
  - SIMPLER solves u, v, p, and p' as a velocity correction only
  - PISO solves for u, v, p', p" where both p' and p" correct both pressure and velocity

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Which Is Better?

- SIMPLE is still used in commercial CFD codes and gets good solutions
- Other methods, although solving an extra equation can take less time
- Hard to distinguish between others
- PISO good when other properties not linked to momentum equtions
- PISO also good for transient problems
- Different relaxation factors in each

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