

Solution of Navier-Stokes Equations. Part Two

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Computational Fluid Dynamics

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Outline

- Review basics of integrating Navier-Stokes and continuity
- Pressure and the staggered grid
- Finite-volume momentum equations
 - Momentum equation coefficients
- Finite-volume continuity
- Finite-volume momentum and continuity combine into an equation for pressure
- Solving the equations

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Review False Diffusion

- Upwind differencing causes errors similar to having a “diffusion” coefficient that is too large
- Causes smearing of results
- Especially noticeable in flows with sharp gradients and shock waves
- Effect is reduced if flow is aligned with grid (not always possible to do this)
- Different from artificial diffusion

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Review Navier-Stokes

- Continuity and momentum equations
- Apply previous work for general variable, ϕ , to finding u , v , and w
- Must consider nonlinearity (e.g., $u\phi$ can be uu , uv , or uw for momentum)
 - Use “outer iteration process”
 - Assume values for u , v , and w
 - Use these values to compute the convection/diffusion coefficients a_E , a_N , etc.
 - Use new u , v , w values to update a_E , a_N , etc.

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Review Navier-Stokes Approach

- Compressible flows
 - Solve continuity and momentum for three velocity components and density
 - Get pressure from equation of state
- Incompressible flows
 - Mach number is low
 - Density is a problem input, often related to temperature (may or may not be constant)
 - Solve continuity and momentum for three velocity components and pressure

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Review Incompressible Flows

- Mach number is low (< 0.3)
- Density (e.g., $\rho = p/RT$) may depend on mean, but not local pressure
- Can have equations like $\rho = \rho_0(1 + \beta T)$ for where $\beta = -(1/\rho)(\partial\rho/\partial T)_p$
- Density may be constant, but need not be for incompressible flows
 - Furnace flows a good example of this
- Basic idea is that density does not depend on local pressure

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Review Steady 2D Problem

- Continuity and momentum equations
 - Have x and y direction convection-diffusion
 - Now have source term and pressure gradient

$$\frac{\partial p u}{\partial x} + \frac{\partial p v}{\partial y} = 0$$

$$\frac{\partial p u u}{\partial x} + \frac{\partial p v u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \mu \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} + S(u)$$

$$\frac{\partial p u v}{\partial x} + \frac{\partial p v v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \mu \frac{\partial v}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial v}{\partial y} + S(v)$$

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Review Problem with Pressure

- Consider the x momentum equation for the u velocity component at point P

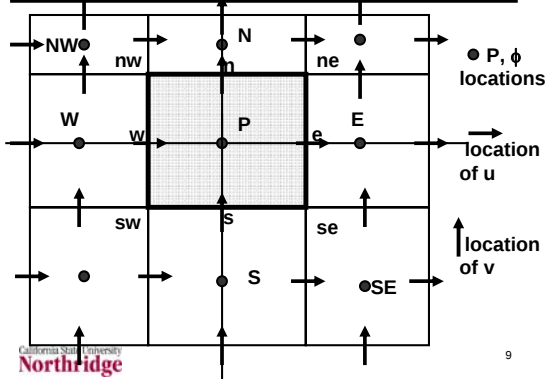
$$\frac{\partial p}{\partial x} \bigg|_P = \frac{p_E - p_W}{2\delta x}$$

- Second-order expression for pressure gradient at P
- With this expression the velocity at P is not affected by the pressure at P
 - Also a checkerboard pattern of pressure would compute as zero pressure gradient

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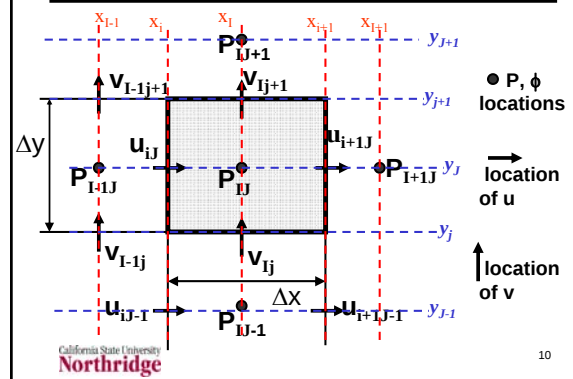
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Review Staggered Grid



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Review Staggered Grid II



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Finite Volume Equations

- Extend previous results for one dimension to two dimensions
- Can use any of the difference methods discussed for convection and diffusion
 - Have four faces, n, e, s, w, in 2D
 - Apply same relations to get coefficients a_N , a_S , a_E , and a_W , using $F = \rho u$ and $D = \Gamma/\delta$ for each face
 - As before $a_P = (a_N + a_S + a_E + a_W + \Delta F)$
 - Here $\Delta F = (F_n - F_s) + (F_e - F_w)$

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Finite Volume Equations II

- Integration of pressure terms

$$\left(\int_{\Delta V} \frac{\partial p}{\partial x} dV \right)_{IJ} \approx \frac{p_{IJ} - p_{I-1,J}}{x_I - x_{I-1}} (x_I - x_{I-1}) A_{IJ} = (p_{IJ} - p_{I-1,J}) A_{IJ} = (p_{IJ} - p_{I-1,J}) (y_{j+1} - y_j) \Delta z$$

$$\left(\int_{\Delta V} \frac{\partial p}{\partial y} dV \right)_{IJ} \approx \frac{p_{IJ} - p_{I,J-1}}{y_J - y_{J-1}} (y_J - y_{J-1}) A_{IJ} = (p_{IJ} - p_{I,J-1}) A_{IJ} = (p_{IJ} - p_{I,J-1}) (x_{i+1} - x_i) \Delta z$$

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Finite Volume Equations III

- Have similar equations for u and v
- b represents integrated source term
- Note that a_k coefficients vary from node to node and are different for u and v

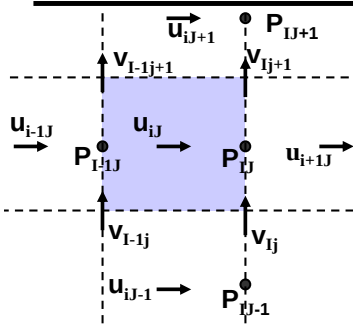
$$a_N^{(u)} u_{iJ+1} + a_S u_{iJ-1} + a_E u_{i+1J} + a_W u_{i-1J} - a_P u_{iJ} = (p_{iJ} - p_{i-1J}) A_{iJ} + b_{iJ}^{(u)}$$

$$a_N^{(v)} v_{iJ+1} + a_S v_{iJ-1} + a_E v_{i+1J} + a_W v_{i-1J} - a_P v_{iJ} = (p_{iJ} - p_{i-1J}) A_{iJ} + b_{iJ}^{(v)}$$

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Control Volume for u (e face)



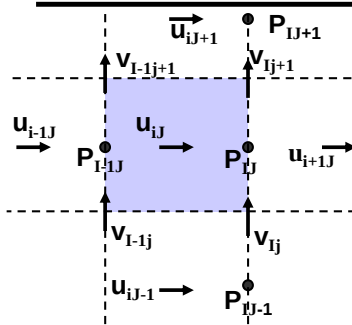
equations for e face at iJ

$$F_e = \frac{F_{i+1J} + F_{iJ}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{iJ} + \rho_{i+1J}}{2} \right) u_{i+1J} + \left(\frac{\rho_{i-1J} + \rho_{iJ}}{2} \right) u_{iJ} \right]$$

$$D_e = \frac{\Gamma_{iJ}}{x_{i+1} - x_i}$$

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Control Volume for u (w face)



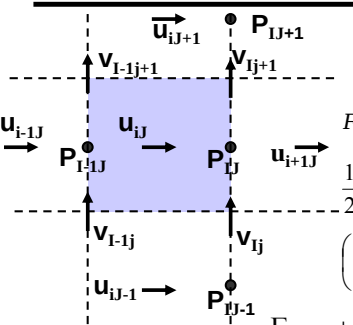
equations for w face at $i-1J$

$$F_w = \frac{F_{i-1J} + F_{iJ}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{i-2J} + \rho_{i-1J}}{2} \right) u_{i-1J} + \left(\frac{\rho_{i-1J} + \rho_{iJ}}{2} \right) u_{iJ} \right]$$

$$D_w = \frac{\Gamma_{i-1J}}{x_i - x_{i-1/2}}$$

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Control Volume for u (n face)



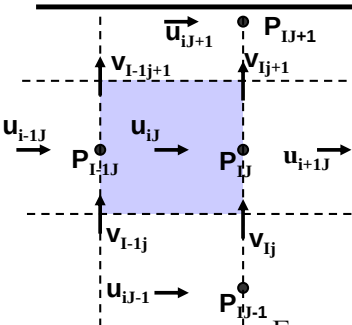
equations for n face at y_{j+1}

$$F_n = \frac{F_{ij+1} + F_{i,j+1}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{iJ} + \rho_{iJ+1}}{2} \right) v_{ij+1} + \left(\frac{\rho_{i-1J} + \rho_{i-1J+1}}{2} \right) v_{i-1,j+1} \right]$$

$$D_n = \frac{\Gamma_{i-1J} + \Gamma_{i-1J+1} + \Gamma_{iJ+1} + \Gamma_{iJ}}{4(y_{j+1} - y_j)}$$

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Control Volume for u (s face)



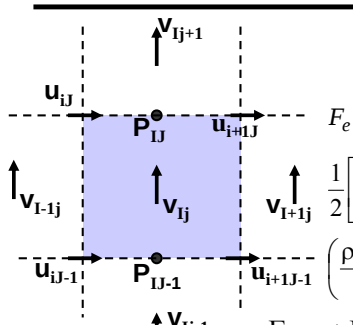
equations for s face at y_j

$$F_s = \frac{F_{ij} + F_{i,j-1}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{iJ} + \rho_{i,j-1}}{2} \right) v_{ij} + \left(\frac{\rho_{i-1J} + \rho_{i-1,j-1}}{2} \right) v_{i-1,j-1} \right]$$

$$D_s = \frac{\Gamma_{i-1J-1} + \Gamma_{i-1J} + \Gamma_{iJ} + \Gamma_{iJ-1}}{4(y_j - y_{j-1})}$$

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Control Volume for v (e face)



equations for e face at x_{i+1}

$$F_e = \frac{F_{i+1J} + F_{i+1J-1}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{iJ} + \rho_{i+1J}}{2} \right) u_{i+1J} + \left(\frac{\rho_{iJ-1} + \rho_{i+1J-1}}{2} \right) u_{i+1J-1} \right]$$

$$D_e = \frac{\Gamma_{iJ+1} + \Gamma_{iJ} + \Gamma_{i+1J} + \Gamma_{i+1J-1}}{4(x_{i+1} - x_i)}$$

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Control Volume for v (w face)

equations for w face at x_i

$$F_w = \frac{F_{ij} + F_{i,j-1}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{i-1,j} + \rho_{ij}}{2} \right) u_{ij} + \left(\frac{\rho_{i-1,j-1} + \rho_{i,j-1}}{2} \right) u_{i,j-1} \right]$$

$$D_w = \frac{\Gamma_{i-1,j-1} + \Gamma_{i-1,j} + \Gamma_{ij} + \Gamma_{i,j-1}}{4(x_i - x_{i-1})}$$

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Control Volume for v (n face)

equations for n face at y_j

$$F_n = \frac{F_{ij+1} + F_{ij}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{i,j+1} + \rho_{ij}}{2} \right) v_{ij+1} + \left(\frac{\rho_{ij} + \rho_{i,j-1}}{2} \right) v_{ij} \right]$$

$$D_n = \frac{\Gamma_{ij}}{y_{j+1} - y_j}$$

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Control Volume for v (s face)

equations for s face at y_{j-1}

$$F_s = \frac{F_{ij} + F_{i,j-1}}{2} = \frac{1}{2} \left[\left(\frac{\rho_{ij} + \rho_{i,j-1}}{2} \right) v_{ij} + \left(\frac{\rho_{i,j-1} + \rho_{i,j-2}}{2} \right) v_{i,j-1} \right]$$

$$D_s = \frac{\Gamma_{i,j-1}}{y_j - y_{j-1}}$$

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Rest Stop

- Now have F and D terms for momentum
 - Can use any method (central, upwind, etc.) to compute a_E , a_N , etc.
- Get finite volume representation of continuity by integration over control volume centered about p_{ij}
- Substitute finite difference momentum equations into finite difference continuity equation to get finite difference equation for pressure

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Rest Stop II

- Solve u momentum equation for u_{ij}
 - Define a_p for this equation as a_{ij}

$$a_N u_{ij+1} + a_S u_{i,j-1} + a_E u_{i+1,j} + a_W u_{i-1,j} - a_P u_{ij} = (p_{ij} - p_{i-1,j}) A_{ij} + b_{ij}^{(u)}$$

$$a_{ij} u_{ij} = a_N u_{ij+1} + a_S u_{i,j-1} + a_E u_{i+1,j} + a_W u_{i-1,j} - (p_{ij} - p_{i-1,j}) A_{ij} - b_{ij}^{(u)}$$

$$u_{ij} = \frac{\sum a_{nb} u_{nb}}{a_{ij}} + (p_{i-1,j} - p_{ij}) \frac{A_{ij}}{a_{ij}} - \frac{b_{ij}^{(u)}}{a_{ij}}$$

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Rest Stop III

- Solve v momentum equation for v_{ij}
 - Define a_p for this equation as a_{ij}

$$a_N v_{ij+1} + a_S v_{i,j-1} + a_E v_{i+1,j} + a_W v_{i-1,j} - a_P v_{ij} = (p_{ij} - p_{i,j-1}) A_{ij} + b_{ij}^{(v)}$$

$$a_P v_{ij} = a_N v_{ij+1} + a_S v_{i,j-1} + a_E v_{i+1,j} + a_W v_{i-1,j} - (p_{ij} - p_{i,j-1}) A_{ij} - b_{ij}^{(v)}$$

$$v_{ij} = \frac{\sum a_{nb} v_{nb}}{a_{ij}} + (p_{i,j-1} - p_{ij}) \frac{A_{ij}}{a_{ij}} - \frac{b_{ij}^{(v)}}{a_{ij}}$$

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Integrated Continuity Equation

$$\int_{\Delta V} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} \right) dV = 0$$

$$\int_{\Delta V} \frac{\partial \rho u}{\partial x} dx dy dz + \int_{\Delta V} \frac{\partial \rho v}{\partial y} dx dy dz = 0$$

$$\delta y \delta z \int_w^e d(\rho u) + \delta x \delta z \int_s^n d(\rho v) = 0$$

$$\delta y \delta z [(\rho u)_e - (\rho u)_w] + \delta x \delta z [(\rho v)_n - (\rho v)_s] = 0$$

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Integrated Continuity Equation II

$$\frac{\delta y \delta z}{\delta x} [(\rho u)_e - (\rho u)_w] + \frac{\delta x \delta z}{\delta y} [(\rho v)_n - (\rho v)_s] = 0$$

$$\frac{\rho_{I+1J} + \rho_{IJ}}{2} (uA)_{i+1J} - \frac{\rho_{IJ} + \rho_{I-1J}}{2} (uA)_{iJ} + \frac{\rho_{IJ+1} + \rho_{IJ}}{2} (vA)_{ij+1} - \frac{\rho_{IJ} + \rho_{IJ-1}}{2} (vA)_{ij} = 0$$

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The Rest is SIMPLE

- Now need to find way to solve for pressures and velocity components
- Use SIMPLE algorithm due to Patankar and Spalding (1971)
 - Semi-Implicit Method for Pressure-Linked Equations
- Started with guessed pressures, p^*
- Solve momentum equations for u^* and v^* based on p^* pressures

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SIMPLE Two

$$u_{ij}^* = \frac{\sum_{nb} a_{nb} u_{nb}^*}{a_{ij}} + (p_{i-1J}^* - p_{IJ}^*) d_{ij} - \frac{b_{ij}^{(u)}}{a_{ij}}$$

$$v_{ij}^* = \frac{\sum_{nb} a_{nb} v_{nb}^*}{a_{ij}} + (p_{IJ-1}^* - p_{IJ}^*) d_{ij} - \frac{b_{ij}^{(v)}}{a_{ij}}$$

- Define correction terms: u' , v' , p'
 - $u = u^* + u'$
 - $v = v^* + v'$
 - $p = p^* + p'$

Now subtract the momentum equations with u^* and v^* from the momentum equations with u and v

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SIMPLE Three

$$u_{ij} = \frac{\sum_{nb} a_{nb} u_{nb}}{a_{ij}} + (p_{i-1J} - p_{IJ}) d_{ij} - \frac{b_{ij}^{(u)}}{a_{ij}}$$

$$u_{ij}^* = \frac{\sum_{nb} a_{nb} u_{nb}^*}{a_{ij}} + (p_{i-1J}^* - p_{IJ}^*) d_{ij} - \frac{b_{ij}^{(u)}}{a_{ij}}$$

$$u_{ij} - u_{ij}^* = \frac{\sum_{nb} a_{nb} (u_{nb} - u_{nb}^*)}{a_{ij}} + [(p_{i-1J} - p_{IJ}) - (p_{i-1J}^* - p_{IJ}^*)] d_{ij}$$

$$u_{ij}' = \frac{\sum_{nb} a_{nb} u_{nb}'}{a_{ij}} + (p_{i-1J}' - p_{IJ}') d_{ij}$$

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SIMPLE Four

$$v_{ij} = \frac{\sum_{nb} a_{nb} v_{nb}}{a_{ij}} + (p_{IJ-1} - p_{IJ}) d_{ij} - \frac{b_{ij}^{(v)}}{a_{ij}}$$

$$v_{ij}^* = \frac{\sum_{nb} a_{nb} v_{nb}^*}{a_{ij}} + (p_{IJ-1}^* - p_{IJ}^*) d_{ij} - \frac{b_{ij}^{(v)}}{a_{ij}}$$

$$v_{ij} - v_{ij}^* = \frac{\sum_{nb} a_{nb} (v_{nb} - v_{nb}^*)}{a_{ij}} + [(p_{IJ-1} - p_{IJ}) - (p_{IJ-1}^* - p_{IJ}^*)] d_{ij}$$

$$v_{ij}' = \frac{\sum_{nb} a_{nb} v_{nb}'}{a_{ij}} + (p_{IJ-1}' - p_{IJ}') d_{ij}$$

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Make it Really SIMPLE

- Since correction terms will vanish as iterations converge, ignore such terms in sums over neighbors!

$$u'_{iJ} = \frac{\sum_{nb} a_{nb} u'_{nb}}{a_{iJ}} + (p'_{I-1J} - p'_{IJ}) d_{iJ} \approx (p'_{I-1J} - p'_{IJ}) d_{iJ}$$

$$v'_{Ij} = \frac{\sum_{nb} a_{nb} v'_{nb}}{a_{Ij}} + (p'_{IJ-1} - p'_{IJ}) d_{Ij} \approx (p'_{IJ-1} - p'_{IJ}) d_{Ij}$$

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Get Ready for Continuity

- Replace u' and v' by $u - u^*$ and $v - v^*$

$$u'_{iJ} = u_{iJ} - u^*_{iJ} = (p'_{I-1J} - p'_{IJ}) d_{iJ} \Rightarrow$$

$$u_{iJ} = u^*_{iJ} + (p'_{I-1J} - p'_{IJ}) d_{iJ}$$

$$v'_{Ij} = v_{Ij} - v^*_{Ij} = (p'_{IJ-1} - p'_{IJ}) d_{Ij} \Rightarrow$$

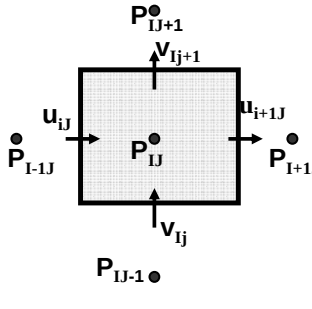
$$v_{Ij} = v^*_{Ij} + (p'_{IJ-1} - p'_{IJ}) d_{Ij} \Rightarrow$$

- Substitute relations between u , v , and p' into finite-volume continuity

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Finite Volume Continuity



$$\frac{\rho_{I+1J} + \rho_{IJ}}{2} (uA)_{i+1J} - \frac{\rho_{IJ} + \rho_{I-1J}}{2} (uA)_{iJ} + \frac{\rho_{IJ+1} + \rho_{IJ}}{2} (vA)_{Ij+1} - \frac{\rho_{IJ} + \rho_{IJ-1}}{2} (vA)_{Ij} = 0$$

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Substitute u , v into Continuity

$$u_{iJ} = u^*_{iJ} + (p'_{I-1J} - p'_{IJ}) d_{iJ} \quad \frac{\rho_{I+1J} + \rho_{IJ}}{2} (uA)_{i+1J} -$$

$$u_{i+1J} = u^*_{i+1J} + (p'_{IJ} - p'_{I+1J}) d_{i+1J} \quad \frac{\rho_{IJ} + \rho_{I-1J}}{2} (uA)_{iJ} +$$

$$v_{Ij} = v^*_{Ij} + (p'_{IJ-1} - p'_{IJ}) d_{Ij} \quad \frac{\rho_{IJ+1} + \rho_{IJ}}{2} (vA)_{Ij+1} -$$

$$v_{Ij+1} = v^*_{Ij+1} + (p'_{IJ} - p'_{IJ+1}) d_{Ij+1} \quad \frac{\rho_{IJ} + \rho_{IJ-1}}{2} (vA)_{Ij} = 0$$

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Finite-Volume Equation for p'

$$\frac{\rho_{I+1J} + \rho_{IJ}}{2} A_{i+1J} [u^*_{i+1J} + (p'_{IJ} - p'_{I+1J}) d_{i+1J}] -$$

$$\frac{\rho_{IJ} + \rho_{I-1J}}{2} A_{iJ} [u^*_{iJ} + (p'_{I-1J} - p'_{IJ}) d_{iJ}] +$$

$$\frac{\rho_{IJ+1} + \rho_{IJ}}{2} A_{Ij+1} [v^*_{Ij+1} + (p'_{IJ} - p'_{IJ+1}) d_{Ij+1}] -$$

$$\frac{\rho_{IJ} + \rho_{IJ-1}}{2} A_{Ij} [v^*_{Ij} + (p'_{IJ-1} - p'_{IJ}) d_{Ij}] = 0$$

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Finite-Volume Equation for p' II

$$\frac{\rho_{I+1J} + \rho_{IJ}}{2} A_{i+1J} u^*_{i+1J} - \frac{\rho_{IJ} + \rho_{I-1J}}{2} A_{iJ} u^*_{iJ} - b_{IJ}$$

$$+ \frac{\rho_{IJ+1} + \rho_{IJ}}{2} A_{Ij+1} v^*_{Ij+1} - \frac{\rho_{IJ} + \rho_{IJ-1}}{2} A_{Ij} v^*_{Ij} +$$

$$\frac{\rho_{I+1J} + \rho_{IJ}}{2} A_{i+1J} d_{i+1J} (p'_{IJ} - p'_{I+1J}) +$$

$$\frac{\rho_{IJ} + \rho_{I-1J}}{2} A_{iJ} d_{iJ} (p'_{IJ} - p'_{I-1J}) +$$

$$\frac{\rho_{IJ+1} + \rho_{IJ}}{2} A_{Ij+1} d_{Ij+1} (p'_{IJ} - p'_{IJ+1}) +$$

$$\frac{\rho_{IJ} + \rho_{IJ-1}}{2} A_{Ij} d_{Ij} (p'_{IJ} - p'_{IJ-1}) = 0$$

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Final Equation for p'

$$a_{I+1J}(p'_{IJ} - p'_{I+1J}) + a_{I-1J}(p'_{IJ} - p'_{I-1J}) + a_{IJ+1}(p'_{IJ} - p'_{IJ+1}) + a_{IJ-1}(p'_{IJ} - p'_{IJ-1}) = b'_{IJ}$$

$$a_{IJ} = a_{I+1J} + a_{I-1J} + a_{IJ+1} + a_{IJ-1}$$

$$a_{I+1J}p'_{I+1J} + a_{I-1J}p'_{I-1J} + a_{IJ+1}p'_{IJ+1} + a_{IJ-1}p'_{IJ-1} - a_{IJ}p'_{IJ} = b'_{IJ}$$

- Solve final equation for p'

Using Corrections

- Use of $p = p^* + p'$ can lead to divergence
 - Use underrelaxation: $p = p^* + \alpha_p p'$, where α_p is the underrelaxation factor ($0 < \alpha_p < 1$)
- Similar underrelaxation factors (between 0 and 1) for velocity correction

$$u_{IJ} = u_{IJ}^* + \alpha_u (p'_{I-1J} - p'_{IJ}) d_{IJ}$$

$$v_{IJ} = v_{IJ}^* + \alpha_v (p'_{IJ-1} - p'_{IJ}) d_{IJ}$$

Putting it All Together

0. Start with initial guesses for p^* , u^* , v^*
1. Start iterations
2. Compute coefficients in finite volume equations for momentum and pressure
3. Do an iteration on u^* and v^* equations
4. Get p' source term
5. Do an iteration on p' equation
6. Use p' to correct p^* , u^* , and v
7. Check convergence; if not converged return to step 1

Conclusion

- Wasn't that simple?
- Next week we will look at another method called SIMPLER (for SIMPLE Revised)
 - It is more complex than SIMPLE
- Will also look at other methods for solving continuity and momentum
 - SIMPLEC
 - PISO