Finite Volume Method for Convection Derivatives

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Computational Fluid Dynamics

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Outline

- Review numerical analysis basics
- Conclude results for diffusion with source analysis
- Introduce finite-volume method for convection
- Note need for average ϕ value at cell faces for convection terms
- Examine use of arithmetic mean as average value and show its problems

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Review Numerical Analysis

- Finite-difference expressions
  - Forward, backward, central
  - Truncation error from Taylor series proportional to step-size, h, to n^{th} power is called n^{th} order error $\varepsilon = O(h^2)$
- Finite difference grids x_i, y_i, z_k, t_n with $u(x_i, y_i, z_k, t_n) = u^n_{ijk}$
- Used Taylor series to get expressions for derivatives on grids
- Greater error on non-uniform grids

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First Derivative Expressions

First order forward $$f_i' = \frac{f_{i+1} - f_i}{h} + O(h)$$

First order backward $$f_i' = \frac{f_i - f_{i-1}}{h} + O(h)$$

Second order central $$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

Second order forward $$f_i' = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} + O\left(h^2\right)$$

Second order backward $$f_i' = \frac{f_{i-2} - 4f_{i-1} + 3f_i}{2h} + O\left(h^2\right)$$

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Other Expressions

Second derivative $$f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2} + O\left(h^2\right)$$

Arithmetic mean $$f_i = \frac{f_{i+1} + f_{i-1}}{2} + O(h^2)$$

Uneven step sizes $h_{i+1} = x_{i+1} - x_i$; $r_i = h_{i+1}/h_i$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} - \frac{f_i''}{2!}\frac{h_{i+1}^2 - h_i^2}{h_{i+1} + h_i} - \frac{f_i'''}{3!}\frac{h_{i+1}^3 + h_i^3}{h_{i+1} + h_i} + \ldots..$$

$$f_i'' = 2\frac{f_{i+1} + r_i f_{i-1} - (1 + r_i)f_i}{h_{i+1}^2 + h_{i+1}h_i} - \frac{f_i'''}{3}\frac{h_{i+1}^2 - h_i^2}{h_{i+1} + h_i} - \frac{f_i''''}{12}\frac{h_{i+1}^3 - h_i^3}{h_{i+1} + h_i} + \ldots.$$

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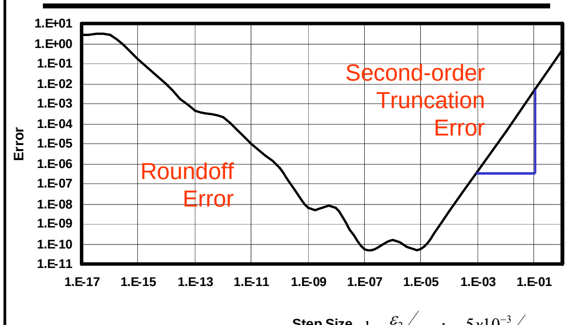
Truncation and Roundoff



$$n = \frac{\log \frac{\varepsilon_2}{\varepsilon_1}}{\log \frac{h_2}{h_1}} = \frac{\log \frac{5x10^{-3}}{5x10^{-7}}}{\log 100} = 2$$

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Finite-volume Approach

- Integrate PDE over a small volume
- Where derivatives occur, replace them by finite-difference expressions
- Where source terms occur, replace terms like $\int_{\Delta V} S dV$ by $S_{avg}\Delta V$
  - S_{avg} is average S for control volume
- Finite Volume grid notation

Variables defined at Nodes

Volume boundaries at Faces

$x_W = x_{i-1}$ $x_w = x_{i-1/2}$ $x_P = x_i$ $x_e = x_{i+1/2}$ $x_E = x_{i+1}$

$\delta x_{WP} = x_P - x_W$, $\delta x_{wP} = x_P - x_w$, $\delta x_{Pe} = x_e - x_P$, $\delta x_{PE} = x_E - x_P$

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Finite Volume Example

- Consider 1D diffusion and source only

$$\frac{d}{dx}\Gamma\frac{d\varphi}{dx} + S = 0 \qquad \int\left(\frac{d}{dx}\Gamma\frac{d\varphi}{dx} + S\right)dV = \int\left(\frac{d}{dx}\Gamma\frac{d\varphi}{dx} + S\right)A dx = 0$$

- Integrate over finite-volume grid

$x_W = x_{i-1}$ $x_w = x_{i-1/2}$ $x_P = x_i$ $x_e = x_{i+1/2}$ $x_E = x_{i+1}$

$\delta x_{WP} = x_P - x_W$, $\delta x_{wP} = x_P - x_w$, $\delta x_{Pe} = x_e - x_P$, $\delta x_{PE} = x_E - x_P$

$$A\int\frac{d}{dx}\Gamma\frac{d\varphi}{dx}dx + \int S dV = A\int_{x_w}^{x_e} d\left(\Gamma\frac{d\varphi}{dx}\right) + \bar{S}\Delta V = \left(\Gamma A\frac{d\varphi}{dx}\right)_e - \left(\Gamma A\frac{d\varphi}{dx}\right)_w + \bar{S}\Delta V = 0$$

- What are Γ and $d\phi/dx$ at w and e faces?

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Finite Volume Example II

- Use mean value for coefficient
  - $\Gamma_e = (\Gamma_P + \Gamma_E)/2$ and $\Gamma_w = (\Gamma_W + \Gamma_P)/2$
- Use central difference for $d\phi/dx$
  - $d\phi/dx|_e = (\phi_E - \phi_P)/\delta x$ $d\phi/dx|_w = (\phi_P - \phi_W)/\delta x$
- Substitute into $\left(\Gamma A\frac{d\varphi}{dx}\right)_e - \left(\Gamma A\frac{d\varphi}{dx}\right)_w + \bar{S}\Delta V = 0$
  - Define $\bar{S}\Delta V = S_u + S_p\varphi_P$

$$\Gamma_e A_e\frac{\varphi_E - \varphi_P}{\delta x_{PE}} - \Gamma_w A_w\frac{\varphi_P - \varphi_W}{\delta x_{WP}} + S_u + S_p\varphi_P = \frac{\Gamma_e A_e\varphi_E}{\delta x_{PE}} + \frac{\Gamma_w A_w\varphi_W}{\delta x_{WP}} + S_u - \left(\frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_w A_w}{\delta x_{WP}} + S_p\right)\varphi_P = 0$$

  - Define $a_E = \frac{\Gamma_e A_e}{\delta x_{PE}}$ $a_W = \frac{\Gamma_w A_w}{\delta x_{WP}}$ $a_P = \frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_w A_w}{\delta x_{WP}} - S_p = a_E + a_W - S_p$

$$a_E\varphi_E + a_W\varphi_W + S_u - a_P\varphi_P = 0 \qquad a_E\varphi_E + a_W\varphi_W - a_P\varphi_P = -S_u$$

$$a_P\varphi_P = a_E\varphi_E + a_W\varphi_W + S_u$$

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Finite-Volume Example III

- This represents equation for one node
  - Need to write equations for all nodes
  - Simple for one-dimensional problem
  - Here is where i, i+1, i-1 notation is easier
    - $a_{W,i}\phi_{i-1} - a_{p,i}\phi_i + a_{E,i}\phi_{i+1} = -S_{u,i}$
    - Apply this result for nodes from i = 1 to i = N – 1
    - Nodes i = 0 and i = N are boundary conditions
  - Example: constant property values:
    - Γ = 1 kg/m·s
    - A = 1 m²
    - ΓA = 1 kg·m/s
    - δx = L/N with L = 1 m

$$a_E = a_W = \frac{\Gamma A}{\delta x} = \frac{\frac{1\,kg}{m\cdot s}\left(1\,m^2\right)}{\frac{1\,m}{N}} = N\frac{kg}{s}$$

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Finite-Volume Example IV

- Consider source term
  - Let S = $\beta\phi$ with β = –10 kg/m³·s
  - Get average source term

$$\bar{S}\Delta V = \int_{\Delta V}\beta\varphi dV = A\beta\int_{\delta x}\varphi dx \approx A\beta\delta x\varphi_P$$

  - Get values for S_u and S_p from definition

$$\bar{S}\Delta V = S_u + S_p\varphi_P$$

$$\bar{S}\Delta V = A\beta\delta x\varphi_P \text{ if } S_u = 0 \text{ and } S_p = A\beta\delta x$$

  - From A = 1 m², δx = (1 m)/N, and β = –10 kg/m³·s, S_p = A$\beta\delta$x = (–10 kg/s) / N

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Finite-Volume Example V

- Get numerical values for all coefficients

$$a_W\varphi_{i-1} - a_P\varphi_i + a_E\varphi_{i+1} = -S_u \qquad S_u = 0 \quad S_p = \frac{-10}{N}\frac{kg}{s}$$

$$a_E = a_W = N\frac{kg}{s} \qquad a_P = a_E + a_W - S_p = \left(2N + \frac{10}{N}\right)\frac{kg}{s}$$

- Substitute into equation and rearrange

$$N\varphi_{i-1} - \left(2N + \frac{10}{N}\right)\varphi_i + N\varphi_{i+1} = 0 \quad \Rightarrow \quad \varphi_{i-1} - \left[2 + \frac{10}{N^2}\right]\varphi_i + \varphi_{i+1} = 0$$

- Need to handle boundary conditions
  - Let ϕ_{left} = 2 at x = 0 and ϕ_{right} = 3 at x = L = 1

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Finite-Volume Example VI

- Nodal values $\phi_0 = \phi_{left} = 2$ and $\phi_N = \phi_{right} = 3$ set by boundary conditions
  – At i = 1 (1 node in from left boundary) $\phi_0 - (2 + 10/N^2)\phi_1 + \phi_2 = 0$ with $\phi_0 = \phi_{left} = 2$ gives $-(2 + 10/N^2)\phi_1 + \phi_2 = -\phi_{left} = -2$
  – At i = N – 1 (1 node in from right boundary) $\phi_{N-2} - (2 + 10/N^2)\phi_{N-1} + \phi_N = 0$ with $\phi_N = \phi_{right} = 3$ gives $\phi_{N-2} - (2 + 10/N^2)\phi_{N-1} = -\phi_{right} = -3$
  – General equation applies at all other nodes
  – Pick N = 5 so $10/N^2 = 10/5^2 = 10/25 = 0.4$ and $2 + 10/N^2 = 2.4$
  – Solve equations for i = 1, 2, 3, and 4

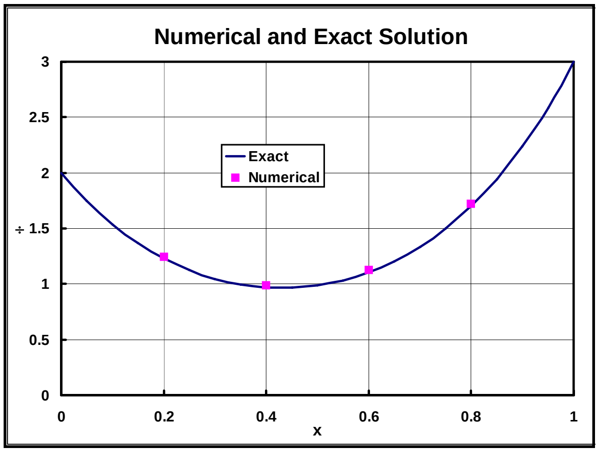
Finite-Volume Example VI

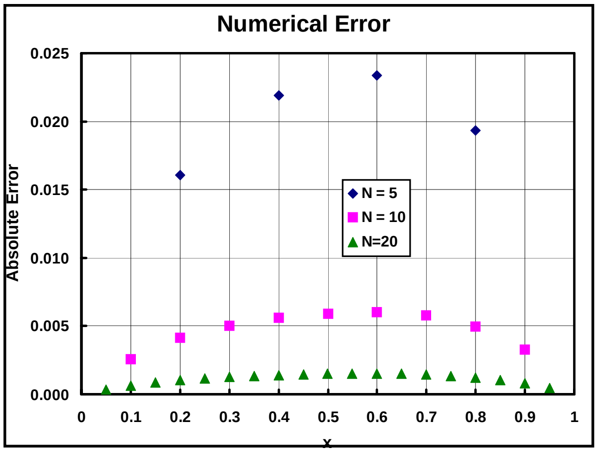
- System of equations to solve

$$\begin{aligned}
-2.4\phi_1 + \phi_2 &= -2 \\
\phi_1 - 2.4\phi_2 + \phi_3 &= 0 \\
\phi_2 - 2.4\phi_3 + \phi_4 &= 0 \\
\phi_3 - 2.4\phi_4 &= -3
\end{aligned}$$

- Matrix form

$$\begin{bmatrix} -2.4 & 1 & 0 & 0 \\ 1 & -2.4 & 1 & 0 \\ 0 & 1 & -2.4 & 1 \\ 0 & 0 & 1 & -2.4 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ -3 \end{bmatrix}$$





Error in Gradient

Exact dφ/dx at x = 0 is -5.54268				
	Second order		First Order	
	dφ/dx	Error	dφ/dx	Error
N = 5	-5.0175	0.5252	-3.7719	1.7708
N = 10	-5.3713	0.1713	-4.6014	0.9413
N = 20	-5.5024	0.0403	-5.0648	0.4779
Exact dφ/dx at x = L is 8.98450				
	Second order		First Order	
	dφ/dx	Error	dφ/dx	Error
N = 5	8.1181	0.8664	6.3976	2.5869
N = 10	8.7104	0.2741	7.5899	1.3946
N = 20	8.9184	0.0661	8.2704	0.7141

Convection Terms

- Steady equation with convection and diffusion terms in one dimension $\frac{d\rho u\varphi}{dx} = \frac{d}{dx}\Gamma\frac{d\varphi}{dx}$
- Steady continuity equation in one dimension $\frac{d\rho u}{dx} = 0$
- Apply finite volume approach to integrate small volume

$$\int \frac{d\rho u\varphi}{dx}dV = \int \frac{d}{dx}\Gamma\frac{d\varphi}{dx}dV \qquad \int \frac{d\rho u}{dx}dV = 0$$

Review Finite-volume Approach

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Integration of PDEs

$$\int \frac{d\rho u}{dx} dV = A\int_{x_w}^{x_e} \frac{d\rho u}{dx} dx = 0 \quad \Rightarrow \quad (\rho u A)_e - (\rho u A)_w = 0$$

$$\int \frac{d\rho u\varphi}{dx} dV = A\int_{x_w}^{x_e} \frac{d\rho u\varphi}{dx} dx = (\rho u\varphi A)_e - (\rho u\varphi A)_w$$

$$=$$

$$\int \frac{d}{dx}\Gamma\frac{d\varphi}{dx} dV = \int_{x_w}^{x_e} \frac{d}{dx}\Gamma\frac{d\varphi}{dx} A dx = \left(\Gamma A\frac{d\varphi}{dx}\right)_e - \left(\Gamma A\frac{d\varphi}{dx}\right)_w$$

$$(\rho u\varphi A)_e - (\rho u\varphi A)_w = \left(\Gamma A\frac{d\varphi}{dx}\right)_e - \left(\Gamma A\frac{d\varphi}{dx}\right)_w$$

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Integration of PDEs II

- Look at constant areas so that area drops out of equation

$$(\rho u\varphi)_e - (\rho u\varphi)_w = \left(\Gamma\frac{d\varphi}{dx}\right)_e - \left(\Gamma\frac{d\varphi}{dx}\right)_w$$

- Use previous results for diffusion terms

$$\Gamma_e = \frac{\Gamma_E + \Gamma_P}{2} \quad \left(\frac{d\varphi}{dx}\right)_e = \frac{\varphi_E - \varphi_P}{\delta x_{PE}} \quad \Gamma_w = \frac{\Gamma_W + \Gamma_P}{2} \quad \left(\frac{d\varphi}{dx}\right)_w = \frac{\varphi_P - \varphi_W}{\delta x_{PW}}$$

- Define F = ρu and D = $\Gamma/\delta x$

$$F_e\varphi_e - F_w\varphi_w = D_e(\varphi_E - \varphi_P) - D_w(\varphi_P - \varphi_E)$$

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Integration of PDEs II

- How do we relate face values to nodal values for convection terms?

$$F_e\varphi_e - F_w\varphi_w = D_e(\varphi_E - \varphi_P) - D_w(\varphi_P - \varphi_E)$$

- First thought: use (second-order accurate) arithmetic mean for face values

$$F_e\frac{\varphi_E + \varphi_P}{2} - F_w\frac{\varphi_P + \varphi_W}{2} = D_e(\varphi_E - \varphi_P) - D_w(\varphi_P - \varphi_W)$$

$$0 = \left(D_e - \frac{F_e}{2}\right)\varphi_E + \left(D_w + \frac{F_w}{2}\right)\varphi_W - \left(D_e + \frac{F_e}{2} + D_w - \frac{F_w}{2}\right)\varphi_P$$

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Integration of PDEs III

- Revise ϕ_P coefficient to include sum of ϕ_W and ϕ_E coefficients

$$0 = \left(D_e - \frac{F_e}{2}\right)\varphi_E + \left(D_w + \frac{F_w}{2}\right)\varphi_W - \left(D_e + \frac{F_e}{2} + D_w - \frac{F_w}{2}\right)\varphi_P$$

$$0 = \quad a_E \quad \varphi_E + \quad a_W \quad \varphi_W - \quad a_P \quad \varphi_P$$

$$a_P = \left(D_e + \frac{F_e}{2} + D_w - \frac{F_w}{2}\right) + (F_e - F_e + F_w - F_w) =$$

Adds zero

$$D_e - \frac{F_e}{2} + D_w + \frac{F_w}{2} + F_e - F_w = a_E + a_W + F_e - F_w = a_P$$

$$0 = a_E\varphi_E + a_E\varphi_W - (a_E + a_W + F_e - F_w)\varphi_P$$

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Example Problem Solution

- Constant ρ, u, and Γ with $\phi = \phi_0$ at x = 0 and $\phi = \phi_L$ at x = L

$$\frac{d\rho u\varphi}{dx} = \frac{d}{dx}\Gamma\frac{d\varphi}{dx} \quad \Rightarrow \quad \frac{\rho u}{\Gamma}\frac{d\varphi}{dx} = \frac{d}{dx}\frac{d\varphi}{dx}$$

$$0 = \frac{d}{dx}\left(\frac{d\varphi}{dx} - \frac{\rho u}{\Gamma}\varphi\right) \quad \Rightarrow \quad \frac{d\varphi}{dx} - \frac{\rho u}{\Gamma}\varphi = C_1$$

$$\varphi = e^{\frac{\rho u x}{\Gamma}}\left[\int C_1 e^{-\frac{\rho u x}{\Gamma}} dx + C_2\right] = e^{\frac{\rho u x}{\Gamma}}\left[C_1\frac{\Gamma}{\rho u}e^{-\frac{\rho u x}{\Gamma}} + C_2\right]$$

$$= \frac{C_1\Gamma}{\rho u} + + C_2 e^{\frac{\rho u x}{\Gamma}}$$

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Example Problem Solution II

- Fit boundary condition $\phi = \phi_0$ at x = 0 and $\phi = \phi_L$ at x = L to get C_1 and C_2

$$\varphi(x) = \frac{C_1\Gamma}{\rho u} + C_2 e^{\frac{\rho u x}{\Gamma}} \Rightarrow \varphi_0 = \varphi(0) = \frac{C_1\Gamma}{\rho u} + C_2$$

$$\varphi_L = \varphi(L) = \frac{C_1\Gamma}{\rho u} + C_2 e^{\frac{\rho u L}{\Gamma}} \Rightarrow \varphi_L - \varphi_0 = C_2 e^{\frac{\rho u L}{\Gamma}} - C_2$$

$$\varphi_0 = \frac{C_1\Gamma}{\rho u} + C_2 = \frac{C_1\Gamma}{\rho u} + \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1} \Rightarrow C_1 = \frac{\rho u}{\Gamma}\left(\varphi_0 - \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1}\right)$$

- Substitute C_1 and C_2 into $\phi(x)$

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Example Problem Solution III

$$\varphi(x) = \frac{C_1\Gamma}{\rho u} + C_2 e^{\frac{\rho u x}{\Gamma}} = \frac{\Gamma}{\rho u}\frac{\rho u}{\Gamma}\left(\varphi_0 - \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1}\right) + \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1} e^{\frac{\rho u x}{\Gamma}}$$

$$\varphi(x) = \varphi_0 - \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1} + \frac{\varphi_L - \varphi_0}{e^{\frac{\rho u L}{\Gamma}} - 1} e^{\frac{\rho u x}{\Gamma}}$$

$$\frac{\varphi(x) - \varphi_0}{\varphi_L - \varphi_0} = -\frac{1}{e^{\frac{\rho u L}{\Gamma}} - 1} + \frac{1}{e^{\frac{\rho u L}{\Gamma}} - 1} e^{\frac{\rho u x}{\Gamma}} = \frac{e^{\frac{\rho u x}{\Gamma}} - 1}{e^{\frac{\rho u L}{\Gamma}} - 1}$$

$$\boxed{\frac{\varphi(x) - \varphi_0}{\varphi_L - \varphi_0} = \frac{e^{\frac{\rho u x}{\Gamma}} - 1}{e^{\frac{\rho u L}{\Gamma}} - 1}}$$

Pe = $\rho uL/\Gamma$

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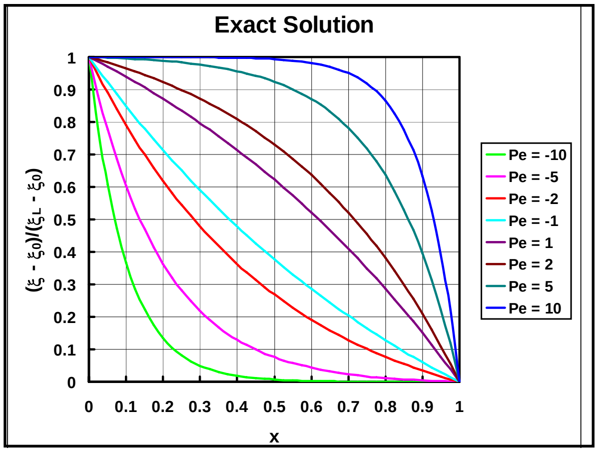

Apply Finite-Volume Equation

$$0 = a_E\varphi_E + a_E\varphi_W - (a_E + a_W + F_e - F_w)\varphi_P$$

- In this example ρu and Γ are constants
  - With ρu constant $F_e = F_w$ in a_p coefficient
- Grid is uniform so δx is constant

$$a_E = D_e - \frac{F_e}{2} = \frac{\Gamma}{\delta x} - \frac{\rho u}{2} \qquad a_W = D_w + \frac{F_w}{2} = \frac{\Gamma}{\delta x} + \frac{\rho u}{2}$$

$$a_W\varphi_{i-1} - a_P\varphi_i + a_E\varphi_E = 0 \qquad a_P = a_E + a_W + F_e - F_w = a_E + a_W$$

- Boundary conditions at x = 0 and x = L different for convection and diffusion

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Boundary Conditions

- At x = 0 (and x = L) we know $\phi = \phi_{left}$ (and $\phi = \phi_{right}$) and do not need mean value equation for ϕ in convection term

$$F_e\frac{\varphi_E + \varphi_P}{2} - F_w\frac{\varphi_P + \varphi_W}{2} = D_e(\varphi_E - \varphi_P) - D_w(\varphi_P - \varphi_W)$$

|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|
left 1 2 3 4 5 6 right

$$F_e\frac{\varphi_2 + \varphi_1}{2} - F_w\varphi_{left} = D_e(\varphi_2 - \varphi_1) - D_{left}(\varphi_1 - \varphi_{left})$$

$$D = \frac{\Gamma}{\delta x} \qquad D_{left} = \frac{\Gamma}{\delta x/2} = \frac{2\Gamma}{\delta x} = 2D$$

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Boundary Conditions II

- Equation for left boundary

$$F\frac{\varphi_2 + \varphi_1}{2} - F\varphi_{left} = D(\varphi_2 - \varphi_1) - 2D(\varphi_1 - \varphi_{left})$$

$$-\left(\frac{F}{2} + 3D\right)\varphi_1 + \left(D - \frac{F}{2}\right)\varphi_2 = -(F + 2D)\varphi_{left}$$

|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|-----●-----|
left 1 2 3 4 5 6 right

- Equation for right boundary

$$F_e\varphi_{right} - F_w\frac{\varphi_P + \varphi_W}{2} = D_{right}(\varphi_{right} - \varphi_P) - D_w(\varphi_P - \varphi_W)$$

D_{right} = 2D

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Boundary Conditions III

$$F\varphi_{right} - F\frac{\varphi_P+\varphi_W}{2} = 2D(\varphi_{right}-\varphi_P) - D(\varphi_P-\varphi_W)$$

$$F\varphi_{right} - F\frac{\varphi_{N-1}+\varphi_{N-2}}{2} = 2D(\varphi_{right}-\varphi_{N-1}) - D(\varphi_{N-1}-\varphi_{N-2})$$

$$\left(D+\frac{F}{2}\right)\varphi_{N-2} - \left(3D-\frac{F}{2}\right)\varphi_{N-1} = -(2D-F)\varphi_{right}$$

- Data: L = 1 m, u = 0.1 m/s, Γ = 0.1 kg/m·s, δx = 0.2 m, ρ = 1 kg/m^3
  - F = ρu = (1 kg/m^3)(0.1 m/s) = 0.1 kg/m^2·s
  - D = $\Gamma/\delta x$ = (0.1 kg/m·s)(0.2 m) = 0.5 kg/m^2·s

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Equations

- Non-boundary equations with $F_e = F_w$, D = 0.5 kg/m^2·s, and F = 0.1 kg/m^2·s

$$a_E\varphi_E + a_E\varphi_W - a_P\varphi_P = 0$$

$$a_E = D - F/2 = 0.45\,kg/m^2\cdot s \qquad a_W = D + F/2 = 0.55\,kg/m^2\cdot s$$

$$a_P = a_E + a_W + F_e - F_w = 0.45\,kg/m^2\cdot s + 0.55\,kg/m^2\cdot s + 0 = 1\,kg/m^2\cdot s$$

- Left-boundary equation

$$-\left(\frac{F}{2}+3D\right)\varphi_1 + \left(D-\frac{F}{2}\right)\varphi_2 = -(F+2D)\varphi_{left}$$

$$-1.55\varphi_1 + 0.45\varphi_2 = -1.1\varphi_{left}$$

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Equations II

- Right-boundary equation

$$\left(D+\frac{F}{2}\right)\varphi_{N-2} - \left(3D-\frac{F}{2}\right)\varphi_{N-1} = -(2D-F)\varphi_{right}$$

- Set D = 0.5 kg/m^2·s, F = 0.1 kg/m^2·s

$$.55\varphi_{N-2} - 1.45\varphi_{N-1} = -0.9\varphi_{right}$$

- Boundary equations for ϕ_{left} = 1 and ϕ_{right} = 0

$$-1.55\varphi_1 + 0.45\varphi_2 = -1.1$$

$$.55\varphi_{N-2} - 1.45\varphi_{N-1} = 0$$

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Equations III

$$\begin{array}{lllllr}
-1.55\phi_1 & +0.45\phi_2 & & & & =-1.1 \\
0.55\phi_1 & -\phi_2 & +0.45\phi_3 & & & =0 \\
 & 0.55\phi_2 & -\phi_3 & +0.45\phi_4 & & =0 \\
 & & 0.55\phi_3 & -\phi_4 & +0.45\phi_5 & =0 \\
 & & & 0.55\phi_4 & -1.45\phi_5 & =0
\end{array}$$

$$\begin{bmatrix} -1.55 & 0.45 & 0 & 0 & 0 \\ 0.55 & -1 & 0.45 & 0 & 0 \\ 0 & 0.55 & -1 & 0.45 & 0 \\ 0 & 0 & 0.55 & -1 & 0.45 \\ 0 & 0 & 0 & 0.55 & -1.45 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} -1.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solution on next chart for F/D = 0.2

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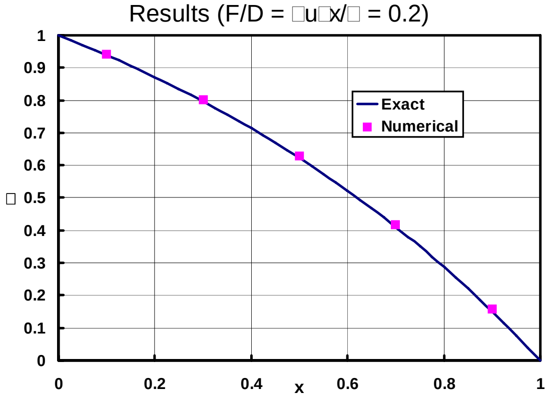
Comparison of Numerical and Exact Results (F/D = □u□x/□ = 0.2)



Changed Inputs

- Recalculate finite-difference coefficients for u = 2.5 m/s as only change
  - D = 0.5 kg/m^2·s, F = 2.5 kg/m^2·s, F/D = 5

$$\begin{bmatrix} -2.75 & -0.75 & 0 & 0 & 0 \\ 1.75 & -1 & -0.75 & 0 & 0 \\ 0 & 1.75 & -1 & -0.75 & 0 \\ 0 & 0 & 1.75 & -1 & -0.75 \\ 0 & 0 & 0 & 1.75 & -0.25 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} -3.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solution on next chart

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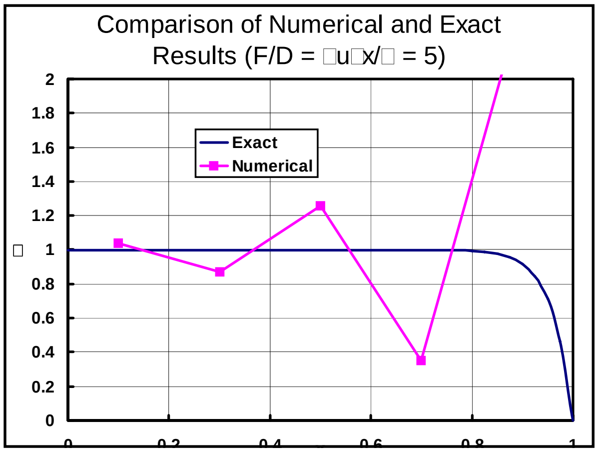

Resolve Grid

- Change δx from 0.2 to 0.05
- Changes D to 2 kg/m^2·s
- F stays at 2.5 kg/m^2·s
- F/D ratio changes to 1.25
  - In first example F/D = 0.2
  - In second example F/D = 5
- F/D ratio = $\rho u \delta x/\Gamma$ is called the cell Peclet number
- Results for δx = 0.05 (F/D = 1.25) on next slide

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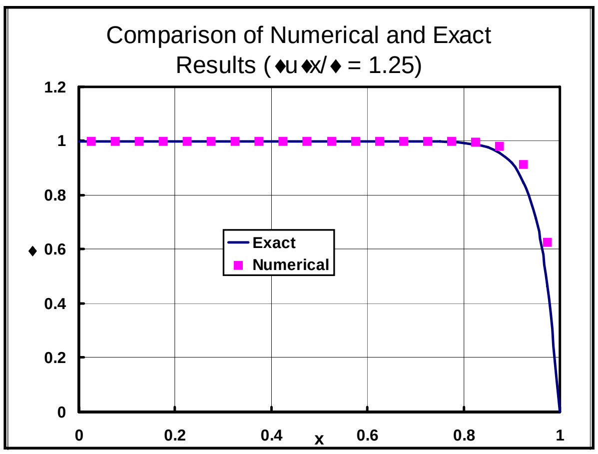

What is Happening Here?

- When flow term F = ρu is large compared to diffusion term D = $\delta x/\Gamma$ the use of average values does not recognize the directionality of the flow
- Keeping the F/D ratio small allows use of average values in convection terms
  - Requires small δx for large flow rates
- How can we use larger δx for large flow rates and keep accuracy?

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