Phase-plane Analysis of Ordinary Differential Equations

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Seminar in Engineering Analysis
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Outline
- Midterm exam two weeks from tonight covering ODEs and Laplace transforms
- Review last class
- Introduction to phase-plane analysis
- Look at two simultaneous ODEs dy₁/dt and dy₂/dt plotted as y₁ vs. y₂
- Look at different “critical points” for different systems of equations

Review Laplace Transforms
- Use transform tables to transform terms in differential equation for y(t) into an algebraic equation for Y(s)
  - Derivative transforms give initial conditions on y(t) and its derivatives
- Manipulate Y(s) equation to sum of individual terms, “Y(s) subterms”, that you can find in transform tables
  - Manipulation may require use of method of partial fractions

Review Partial Fractions
- Method to convert fraction with several factors in denominator into sum of individual factors (in denominator)
  - Example is $F(s) = 1/(s+a)(s+b)$
  - Write $1/(s+a)(s+b) = A/(s+a) + B/(s+b)$
  - Multiply by $(s+a)(s+b)$ and equate coefficients of like powers of s
    - $1 = A(s + b) + B(s + a)$
    - $A + B = 0$ for $s^1$ terms and $1 = bA + aB$ for $s^0$ terms

Review Partial Fractions II
- $A + B = 0$ for $s^1$ terms and $1 = bA + aB$ for $s^0$ terms
- Solving for A and B gives $A = -B = 1/(b - a)$
  - Result: $1/(s+a)(s+b) =$
    - $1/[(b - a)(s + a)] - 1/[(b - a)(s + b)]$
    - So $f(t) = [e^{-at} - e^{-bt}]/(b - a)$
- This actually matches a table entry
- Follow same basic process for more complex fractions
- Special rules for repeated factors and complex factors

Review Partial Fraction Rules
- Repeated fractions for repeated factors
  - $1/[(s+a)^n] = \cdots + A_n/(s+a) + \cdots + A_1/(s+a) + A_0/(s+a)^n$
- Complex factors
  - $1/(s + \alpha - i\beta)(s + \alpha + i\beta) = \cdots + A_0 + B_0/(s + \alpha)^2 + \cdots$
- Pure imaginary factor
  - $1/[(s+i\beta)(s+i\beta)] = \cdots + A_0 + B_0/(s+i\beta)^2 + \cdots$
- Real squared factors
  - $1/[(s-\beta)(s-\beta)] = \cdots + A_0 + B_0/(s-\beta)^2 + \cdots$
Review Systems of ODEs

- Apply Laplace transforms to systems of equations by transforming all ODEs
  - Transform ODE terms like $y_k$ to $Y_k(s)$, $\frac{dy_k}{dt}$ to $sY_k(s) - y_k(0)$, etc.
- Transform all ODEs in system then use Gaussian elimination to get an equation for only one $Y_k(s)$
- Get inverse transform from $Y_k(s)$ to $y_k(t)$
- Repeat for all ODEs

Review Group Exercise

- Solve $y'' - 9y = e^t$ with $y(0) = 0$ and $y'(0) = 2$ by Laplace transforms
- Transform differential equation:
  $$s^2Y(s) - sy(0) - y'(0) - 9Y(s) = \frac{1}{s + 1}$$
- Substituting initial conditions and solve result for $Y(s)$
  $$s^2Y(s) - 0 - 2 - 9Y(s) = \frac{1}{s + 1}$$
  $$Y(s) = \frac{2 + 1/(s + 1)}{s^2 - 9}$$

Review Group Exercise II

$$(s^2 - 9)Y(s) = 2 + 1/(s + 1)$$

- Use partial fractions for last term
  $$\frac{1}{(s^2 - 9)(s + 1)} = \frac{A}{s + 3} + \frac{B}{s + 1} + \frac{C}{s + 1}$$
  $$1 = A(s + 1)(s + 3) + B(s + 1)(s - 3) + C(s^2 - 9)$$
- Set sums of like powers to zero

Review Group Exercise III

$$1 = A(s + 1)(s + 3) + B(s + 1)(s - 3) + C(s^2 - 9)$$

- $s^2$ terms:
  $$0 = A + B + C$$

- $s^1$ terms:
  $$0 = 4A - 2B$$

- $s^0$ terms:
  $$1 = 3A - 3B - 9C$$

- $s^1$ equation gives $B = 2A$
- Substituting $B = 2A$ into $s^2$ equation gives $A + 2A + C = 0$ or $C = \frac{-3A}{2}$
- Substitute $B = 2A$ and $C = \frac{-3A}{2}$ into $s^0$ equation to get $1 = 3A - 3(2A) - 9(-3A)$

Review Group Exercise IV

- From $A = 1/24$ and $B = 2A$: $B = 2/24$ on 7
- From $A = 1/24$ and $C = -3A$: $C = -3/24$

$$Y(s) = \frac{2}{s^2 - 9} + \frac{A}{s + 3} + \frac{B}{s + 1} + \frac{C}{s + 1}$$

- From transform table
  $$y(t) = \frac{2}{3} \sinh(3t) + \frac{1}{24} [e^{3t} + 2e^{-3t} - 3e^{-t}]$$
  $$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Basic Phase Plane Equations

- Look at solution of system of two first-order autonomous (no t dependence) homogenous equations
- Can be single second order equation written as two first order equations:
  $$\frac{d^2y}{dt^2} + \frac{c}{m} \frac{dy}{dt} + \frac{k}{m} y = 0$$
  $$\frac{dv}{dt} + \frac{c}{m} v + \frac{k}{m} y = 0$$
  $$\frac{dy}{dt} - v = 0$$
Phase Plane Analysis

- Look at solutions of systems of equations, here use two equations as an example
- Find certain points, called critical points, that have particular behavior depending on the eigenvalues of the ODE’s
- This leads to a discussion of stability; will a solution tend to zero or increase without bound?

What is a Stable Solution?

- Plot trajectories (plot one dependent variable against the other)
  - For 2 ODE’s dy/dt and dv/dt, plot y vs. v
- If at one time the trajectory is within a distance ε of a point P₀ and for all future times it remains within a distance δ of P₀, the solution is stable
- The solution is unstable if it is not stable
- Want to find criteria for stable solutions

Undamped Vibrations Example

- Equation: d²x/dt² + ω²x = 0 (ω² = k/m)
  - Solution: x = (v₀/ω)sin ωt + x₀ cos ωt
  - As system of equations dx/dt = v and dv/dt = d²x/dt² = −ω²x
  - Define y₁ = x and y₂ = v to get system of equations as dy₁/dt = y₂ and dy₂/dt = −ω²y₁
    
    \[
    \begin{align*}
    y_1' &= a_{11}y_1 + a_{12}y_2 = y_2 \\
    y_2' &= a_{21}y_1 + a_{22}y_2 = -\omega^2 y_1
    \end{align*}
    \]
    
    \[
    \begin{align*}
    a_{11} &= a_{22} = 0 \\
    a_{12} &= 1 \\
    a_{21} &= -\omega^2
    \end{align*}
    \]

Phase Plane Introduction

- Usual plot shows solutions for x = y₁ and v = dx/dt = y₂ as a function of time

Phase Plane Introduction II

- Phase plane plot shows y₂ as a function of y₁ with t as a parameter
- Same as previous plot y₁ is displacement and y₂ is velocity
  - Time differs along plot

Phase Plane Introduction III

- Initial point (t = 0) repeats periodically
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### General Form

- Write as matrix equation $\frac{dy}{dt} = Ay$
- General form for two equations and solutions in terms of $Ax = \lambda x$

  \[
  \begin{align*}
  \frac{dy}{dt} &= Ay \\
  y &= C_1x_1(\e^{-\lambda_1 t}) + C_2x_2(\e^{-\lambda_2 t})
  \end{align*}
  \]

### Stable Solution Criteria

- Look at the system of two equations $\frac{dy}{dt} = Ay$
  - Autonomous systems (no $t$ dependence)
  - We will show that stability depends on the trace of $A = a_{11} + a_{22} = \lambda_1 + \lambda_2$, and the determinant $a_{11}a_{22} - a_{12}a_{21}$ and the discriminant, $\Delta = (\text{Trace } A)^2 - 4 \det A$
  - Review eigenvalues for $2 \times 2$ matrix from September 12 lecture

### Two-by-two Matrix Eigenvalues

- Quadratic equation with two roots for eigenvalues

  \[
  \begin{align*}
  (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0
  \end{align*}
  \]

  \[
  \lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2}
  \]

### Return to Previous Example

- For undamped vibrations we had

  \[
  \begin{align*}
  y_1' &= a_{11}y_1 + a_{12}y_2 = y_1 \\
  y_2' &= a_{21}y_1 + a_{22}y_2 = -\omega^2 y_1 \\
  a_{11} &= a_{22} = 0 \\
  a_{12} &= 1 \\
  a_{21} &= -\omega^2
  \end{align*}
  \]

- This gives trajectory slope as

  \[
  \begin{align*}
  \frac{dy_2}{dy_1} &= \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} = \frac{-\omega^2 y_1 + 0 y_2}{0 y_1 + (1)y_2} = -\frac{\omega^2 y_1}{y_2}
  \end{align*}
  \]

### What Happens if $y_1 = y_2 = 0$?

- Autonomous system of two equations

  \[
  \begin{align*}
  \frac{dy_2}{dt} &= a_{21}y_1 + a_{22}y_2 \\
  \frac{dy_1}{dt} &= a_{11}y_1 + a_{12}y_2
  \end{align*}
  \]

  - Trajectory slope, $dy_2/dy_1$, depends on values of $A$ and may be indeterminate at $y_1 = y_2 = 0$
  - $y_1 = y_2 = 0$ is called a critical point

- For multidimensional systems $y = 0$

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Types of Critical Points

- Critical points are points on a $y_1$-$y_2$ plot that are classified depending on the trajectory shapes at or near these points
- Centers
- Improper Nodes
- Proper Nodes
- Saddle Points
- Spiral Points

Our First Critical Point

- The undamped vibrations solution is an ellipse that does not go through $y_1 = y_2 = 0$
- This type of critical point is called a center

Improper Node

- All trajectories, except two of them have the same limiting direction of the tangents
- The two exceptions will have a different direction

Proper Node

- $x' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$
- $x = c_1 [1] e^t + c_2 [0] e^{4t}$

Unstable Saddle Point

$y' = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} y$

Stable Saddle Point

$y' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} y$
**Spiral Points**

- Asymptotically Unstable
- Asymptotically Stable

![Spiral Points Diagram](http://tutorial.math.lamar.edu/Classes/DE/RepeatedEigenvalues_files/image002.gif)

**Unstable Spiral Source**

\[ y' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} y \]

![Unstable Spiral Source Diagram](http://systems-sciences.uni-graz.at/etextbook/assets/img/img_phpl/s12.png)