Use of Lagrange interpolation to create table for a given function *(see also Example 4 from the textbook, page 111)*

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Suppose a table is to be prepared for function $f(x) = \exp(x)$ on the interval $[0, 1]$. What should the step size h be for the quadratic interpolation (i.e., with the Lagrange polynomial of degree 2) to give an absolute error of at most 10^{-6}.

Solution

Let $x_0, x_1, \ldots$ be the numbers from $[0, 1]$ at which f is evaluated and $x \in [0, 1]$. Suppose that $j \geq 0$ satisfies $x_j \leq x \leq x_{j+2}$, where $x_j = jh$ ($h > 0$). We have for some $\xi(x) \in [0, 1]$,

$$|f(x) - P_2(x)| = \left|\frac{f^{(3)}(\xi(x))}{3!}(x - x_j)(x - x_{j+1})(x - x_{j+2})\right| = \frac{|f^{(3)}(\xi(x))|}{3!}|(x - x_j)(x - x_{j+1})(x - x_{j+2})|, \quad (1)$$

where $P_2(x) = f(x_j)L_{2,j}(x) + f(x_{j+1})L_{2,j+1}(x) + f(x_{j+2})L_{2,j+2}(x)$ and

$$L_{2,j}(x) = \frac{(x - x_{j+1})(x - x_{j+2})}{(x_j - x_{j+1})(x_j - x_{j+2})}, \quad L_{2,j+1}(x) = \frac{(x - x_j)(x - x_{j+2})}{(x_{j+1} - x_j)(x_j - x_{j+1})}, \quad L_{2,j+2}(x) = \frac{(x - x_j)(x - x_{j+1})}{(x_{j+2} - x_j)(x_j - x_{j+1})}.$$

Hence,

$$\begin{aligned} |f(x) - P(x)| &\leq \frac{1}{6}\left(\max_{\xi \in [0,1]} e^{\xi}\right)\left(\max_{x_j \leq x \leq x_{j+2}} |(x - x_j)(x - x_{j+1})(x - x_{j+2})|\right) \\ &\leq \frac{e}{6} \max_{x_j \leq x \leq x_{j+2}} |(x - x_j)(x - x_{j+1})(x - x_{j+2})| \end{aligned}$$

In order to compute the maximum of the expression,

$$|(x - x_j)(x - x_{j+1})(x - x_{j+2})|, \quad (2)$$

on the interval $[x_j, x_{j+2}]$ we find the zeros of the derivative of the function

$$g(x) = (x - x_j)(x - x_{j+1})(x - x_{j+2}), \qquad g'(x) = 3x^2 - 6xhj - 6xh + 3h^2j^2 + 6h^2j + 2h^2.$$

They are

$$h\left(j + 1 + \frac{\sqrt{3}}{3}\right) \quad \text{and} \quad h\left(j + 1 - \frac{\sqrt{3}}{3}\right)$$

Now,

$$g\left(h\left(j + 1 - \frac{\sqrt{3}}{3}\right)\right) = \frac{2h^3\sqrt{3}}{9}, \qquad g\left(h\left(j + 1 + \frac{\sqrt{3}}{3}\right)\right) = -\frac{2h^3\sqrt{3}}{9};$$

thus

$$\max_{x_j \leq x \leq x_{j+2}} |(x - x_j)(x - x_{j+1})(x - x_{j+2})| = \frac{2h^3\sqrt{3}}{9}, \quad (3)$$

and

$$|f(x) - P(x)| \leq \frac{e}{6} \cdot \frac{2h^3\sqrt{3}}{9} = \frac{eh^3\sqrt{3}}{27} \leq 10^{-6}.$$

The last inequality is satisfied for

$$0 < h < \left(\frac{27}{e(\sqrt{3})10^6}\right)^{1/3} = \frac{3^{5/6}}{100e^{1/3}} \approx 0.01789930706.$$

Note: *Computation of the maximum of the expression*

$$|(x - x_j)(x - x_{j+1})(x - x_{j+2})|$$

on the interval $[x_j, x_{j+2}]$ *can be simplified if we observe that this maximum is not affected by a shift. Thus we can choose, for example,* $x_j = -h$, $x_{j+1} = 0$, *and* $x_{j+2} = h$. *The extrema of function* $g(x) = (x - h)x(x + h)$ *on the interval* $[-h, h]$ *are attained at* $x = \pm h/\sqrt{3}$, *and as before, the maximum value of* $|g(x)|$ *is given in* (3).