# Homework 5

Math. 481a, Spring 2026

**SHOW ALL YOUR WORK**

***IMPORTANT*: Please do all your work in space provided.**
**If needed, you can use backspaces. No additional sheets of paper will be accepted.**
**Check that your homework has a total of 8 pages— there are no blank pages.**

**Problem 1.** (3 points)
The following Second Derivative Midpoint Formula

$$f''(x_0) \approx .\frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right]$$

is $O(h^2)$.. Assume that $f''(x_0)$ exists and $f$ is defined in some open interval containing $x_0$. Show that

$$\lim_{h\to 0} \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right] = f''(x_0).$$

**Problem 2.** (4 points)

Approximate the integral $\int_1^{1.5} x^2 \ln x \, dx$ using the Trapezoidal and Simpson's rule. Find a bound for the error in the Trapezoidal and Simpson's rules and compare to the actual errors.

**Problem 3.** (4 points)

Derive Simpson's rule with error term by using

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi).$$

Find $a_0$, $a_1$, and $a_2$ from the fact that Simpson's rule is exact for $f(x) = x^n$ when $n = 1, 2, 3$. Then find $k$ by applying the integration formula with $f(x) = x^4$.

**Problem 4.** (4 points)

Show that a quadrature formula has degree of precision $n$ precisely when $E(x^k) = 0$ for all $k = 0, 1, \ldots, n$ and $E(x^{n+1}) \neq 0$. (See Definition 4.1 on page 195)

**Problem 5.** (2+2+2 points)

Determine the values of $n$ and $h$ required to approximate

$$\int_1^2 x \ln x \, dx$$

to within $10^{-5}$,

(a) using the Composite Trapezoidal rule,
(b) using the Composite Simpson's rule,
(c) using the Composite Midpoint rule.

**Problem 6.** (3 points)

It can be shown that for $f \in C^\infty[a,b]$ the *composite Trapezoidal rule* can be written with an error term in the form

$$\int_a^b f(x)\,dx = h\left[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right] + K_1h^2 + K_2h^4 + K_3h^6 + \cdots, \qquad (1)$$

where $h = (b-a)/n$ and $x_j = a + jh$, $n = 1, 2, \ldots$, with the notation that for $n = 1$ the expression in the bracket on the right hand side of (1) is $\left[\frac{1}{2}f(a) + \frac{1}{2}f(b)\right]$.

The *Romberg integration* uses the *composite Trapezoidal rule* (1) and *Richardson's extrapolation* to approximate $\int_a^b f(x)\,dx$. In particular, the results of the *composite Trapezoidal rule* with $n = 1$: $R_{1,1}$ and and $n = 2$: $R_{2,1}$ are used to compute $O(h^4)$ approximation:

$$R_{2,2} = R_{2.1} + \frac{1}{3}\left(R_{2,1} - R_{1,1}\right) \qquad \text{(Richardson's extrapolation, see page 185 of 10th ed. of the textbook)}$$

Write down explicitly $R_{2,2}$ and recognize it as a well known quadrature formula.

**Problem 7.** (3 points)

Theorem 5.4 (from the textbook) states

*Suppose that $D = \{(t, y) : a \leq t \leq b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $y$, then the initial-value problem*

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

*has unique solution $y(t)$ for $a \leq t \leq b$.*

Consider the initial value problem

$$y' = [\exp(t)y]^2, \quad y(0) = -1, \quad 0 \leq t \leq 1. \tag{2}$$

It is easy to check (the equation is separable) that

$$y(t) = -\frac{2}{1 + \exp(2t)}.$$

is a solution to the initial value problem (2)

Can Theorem 5.4 with the domain $D = \{(t, y) : 0 \leq t \leq 1, -\infty < y < \infty\}$ be applied to (2)?

**Provide details!**

**Problem 8.** (3 points)

Consider the initial value problem

$$y' = t(y+1), \quad y(0) = 0, \quad 0 \le t \le 1. \tag{3}$$

Use Euler's Method with $h = \frac{1}{3}$ to approximate the solution of (3).

**Express your results as proper fractions.**