

## Homework 3

Math. 481a, Spring 2026

**SHOW ALL YOUR WORK**

***IMPORTANT:*** Please do all your work in space provided.

If needed, you can use backspaces. No additional sheets of paper will be accepted.

Check that your homework has a total of 5 pages.

**Problem 1.** (3+3 points)

Consider the nonlinear system:

$$x_1^2 - x_2 - 37 = 0,$$

$$x_1 - x_2^2 - 5 = 0,$$

$$x_1 + x_2 + x_3 - 3 = 0$$

(a) Use Newton's method to obtain the explicit expression for  $\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{J}(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$ , where  $\mathbf{x} = (x_1, x_2, x_3)$ . *Make sure to write the matrix  $\mathbf{J}(\mathbf{x})^{-1}$ .*

(b) Use the result of part (b) to compute  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  with  $\mathbf{x}^{(0)} = (0, 0, 0)$ .

**Problem 2.** (6 points)

Apply Newton's method to the equation  $ax = b$ , with  $a \neq 0$  and arbitrary initial point  $p_0$ . What does  $p_1$  represent? What is  $p_2$  and what is  $p_n$ ,  $n \geq 1$ ?

**Problem 3.** (7 points)

Show that a function  $\mathbf{F} = (f_1, f_2, \dots, f_n) : D \rightarrow \mathbb{R}^n$ , with  $D = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$  and  $a_i \leq b_i$ ,  $i = 1, 2, \dots, n$  is continuous at  $\mathbf{x}_0 \in D$  if and only if for any  $\epsilon > 0$  there exists  $\delta > 0$  such that whenever  $\mathbf{x} \in D$  and  $\|\mathbf{x} - \mathbf{x}_0\|_\infty < \delta$  then  $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)\|_\infty < \epsilon$ . Here the norm  $\|\cdot\|_\infty$  for  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  is defined by  $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$ . **Hint:** Start with Definition 10.3.

**Problem 4.** (3+3 points)

The nonlinear system

$$x_1^2 - 8x_1 + x_2^2 + 6 = 0, \quad x_1x_2^2 + x_1 - 8x_2 + 6 = 0$$

can be transformed into the fixed-point problem

$$x_1 = g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 6}{8}, \quad x_2 = g_2(x_1, x_2) = \frac{x_1x_2^2 + x_1 + 6}{8}.$$

(a) Use Theorem 10.6 to show that  $\mathbf{G} = (g_1, g_2)$  defined on  $D = \{(x_1, x_2) : 0 \leq x_1 \leq 4/3, 0 \leq x_2 \leq 4/3\}$  has a unique fixed point in  $D$ . **Note:** *Make sure to check that all the assumptions of Theorem 10.6 are satisfied.*

(b) Use fixed-iteration (starting with  $\mathbf{x}_0 = (0, 0)$ ) to approximate the solution to within  $10^{-5}$  using  $\|\cdot\|_\infty$ . In other words, stop the iterations when  $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty < 10^{-5}$ . Here, for  $\mathbf{x} = (x_1, x_2)$ ,  $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq 2} |x_i|$ .