Math. 481a, Spring 2026

## Solutions to various problems

### Problem 11, section 3.3

*Suppose that $P(x)$ is a polynomial of unspecified degree for which $P(0) = 2$, P(1)=-1, P(2)=4, and all third-order forward differences are 1. Find the coefficient of $x^2$ in $P(x)$.*

*Solution*:

*First method*

Since $h = 1$, $x_0 = 0$, and $x = x_0 + sh = s$ in Newton's forward difference formula, we have

$$P(0) = 2, \Delta P(0) = P(1) - P(0) = -3,$$
$$\Delta^2 P(0) = (P(2) - P(1)) - (P(1) - P(0)) = 5 - (-3) = 8,$$

and

$$\Delta^3 P(0) = 1, \quad \text{and} \quad \Delta^4 P(0) = \Delta^5 P(0) = \cdots = 0.$$

Next, Newton's forward difference formula gives

$$\begin{aligned}P(x) &= P(0) + x\Delta P(0) + \frac{1}{2}x(x-1)\Delta^2 P(0) + \frac{1}{6}x(x-1)(x-2)\Delta^2 P(0)\\ &=2 - 3x + 4x(x-1) + \frac{1}{6}x(x-1)(x-2)\\ &=2 - \frac{20}{3}x + \frac{7}{2}x^2 + \frac{1}{6}x^3.\end{aligned}$$

The coefficient of $x^2$ is $7/2 = 3.5$.

*Second method*

Since all the third-order forward differences are 1, all fourth-order forward differences are zero. In particular, (since $h = 1$), $f[x_0, x_1, x_2, x_3, x_4] = \frac{1}{4!1^4}\Delta^4 f(x_0) = \frac{1}{24} \cdot 0 = 0$. This also means that $P(x) = Ax^3 + Bx^2 + Cx + D$. Furthermore, since $f[x_0, x_1, x_2, x_3] = \frac{1}{3!1^3}\Delta^3 f(x_0) = \frac{1}{6} \cdot 1 = \frac{1}{6}$, we have $A = \frac{1}{6}$.

$$\begin{aligned}2 = P(0) &= \frac{1}{6} \cdot 0^3 + B \cdot 0^2 + C \cdot 0 + D\\ -1 = P(1) &= \frac{1}{6} + B + C + D\\ 4 = P(2) &= \frac{8}{6} + 4B + 2C + D\end{aligned}$$

Solving the above system we obtain, $B = 7/2$, $C = -\frac{20}{3}$, $D = 2$. The coefficient of $x^2$ is $7/2 = 3.5$.

### Problem 12, section 3.3

*Suppose that $P(x)$ is a polynomial of unspecified degree for which $P(0) = 4$, $P(1) = 9$, $P(2) = 15$ , $P(3) = 18$ and all fourth-order forward differences are 1. Find the coefficient of $x^3$ in $P(x)$.*

*Solution*:

Since all the fourth-order forward differences are 1, all fifth-order forward differences are zero. In particular, (since $h = 1$), $f[x_0, x_1, x_2, x_3, x_4, x_5] = \frac{1}{5!1^5}\Delta^5(x_0) = \frac{1}{120} \cdot 0 = 0$. This also means that $P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$. Furthermore, since $f[x_0, x_1, x_2, x_3, x_4] = \frac{1}{4!1^4}\Delta^4 f(x_0) = \frac{1}{24} \cdot 1 = \frac{1}{24}$, we have $A = \frac{1}{24}$.

$$
\begin{aligned}
4 = P(0) &= \frac{1}{24}\cdot 0^4 + B\cdot 0^3 + C\cdot 0^2 + D\cdot 0^1 + E\\
9 = P(1) &= \frac{1}{24} + B + C + D + E\\
15 = P(2) &= \frac{1}{24}\cdot 16 + 8B + 4C + 2D + E\\
18 = P(3) &= \frac{1}{24}\cdot 81 + 27B + 9C + 3D + E
\end{aligned}
$$

Solving the above system we obtain, $B = -11/12, C = 71/24, D = 35/12, E = 4$. Thus, the coefficient of $x^3$ is $-11/12$.

**Problem 15, section 3.3**

*Find* $\Delta^2 P(10)$ *for the fouth-degree polynomial* $P(x)$ *given* $\Delta^4 P(0) = 24$, $\Delta^3 P(0) = 6$, *and* $\Delta^2 P(0) = 0$, *where* $\Delta P(x) = P(x+1) - P(x)$.

*Solution*: Since $h = 1$, $x_0 = 0$, $x = x_0 + sh = s$, and $P(x)$ is a polynomial of degree 4, Newton's forward difference formula (see, for example, formula (7) of http://www.csun.edu/~hcmth008/481a/481a_n_3_2.pdf) gives

$$
\begin{aligned}
P(s) &= \binom{s}{0}P(0) + \binom{s}{1}\Delta P(0) + \binom{s}{2}\Delta^2 P(0) + \binom{s}{3}\Delta^3 P(0) + \binom{s}{4}\Delta^4 P(0)\\
&= P(0) + s\Delta P(0) + \frac{1}{2}s(s-1)\Delta^2 P(0) + \frac{1}{6}s(s-1)(s-2)\Delta^3 P(0) + \frac{1}{24}s(s-1)(s-2)(s-3)\Delta^4 P(0)\\
&= P(0) + s\Delta P(0) + s(s-1)(s-2) + s(s-1)(s-2)(s-3).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P(10) &= P(0) + 10\Delta P(0) + 10\cdot 9\cdot 8 + 10\cdot 9\cdot 8\cdot 7 = P(0) + \Delta P(0) + 5760,\\
P(11) &= P(0) + 11\Delta P(0) + 8910,\\
P(12) &= P(0) + 12\Delta P(0) + 13200.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta P(11) = P(12) - P(11) = \Delta P(0) + 4290,\\
\Delta P(10) = P(11) - P(10) = \Delta P(0) + 3150
\end{aligned}
$$

and

$$
\Delta^2 P(10) = \Delta P(11) - \Delta P(10) = 1140.
$$

**Problem 21, section 3.3**

*Given*

$$
\begin{aligned}
P_n(x) = f[x_0] &+ f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1)\\
&+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots\\
&+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}
$$

*use* $P_n(x_2)$ *to show that* $a_2 = f[x_0, x_1, x_2]$.

*Solution*: Substituting $x = x_2$, gives

$$P_n(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2).$$

Since

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{and} \quad f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

after some algebra we obtain that

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2].$$

## Problem 1a, section 4.1

*Use the forward-difference and backward-difference formulas to approximations of $f'(0.5)$, $f'(0.6)$, and $f'(0.7)$, if $f(0.5) = 0.4794$, $f(0.6) = 0.5646$, and $f(0.7) = 0.6442$.*

*Solution*:

- Possible aproximations of $f'(0.5)$:

  $$f'(0.5) \approx \frac{f(0.6) - f(0.5)}{0.1} \qquad \text{(two-point forward-difference formula)}$$

  $$f'(0.5) \approx \frac{-3f(0.5) + 4f(0.6) - \frac{1}{2}f(0.7))}{2 \cdot 0.1} \qquad \text{(three-point forward-difference formula)}$$

  No backward-difference formulas can be used to approximate $f'(0.5)$.
- Possible aproximations of $f'(0.6)$:

  $$f'(0.6) \approx \frac{f(0.7) - f(0.6)}{0.1} \qquad \text{(two-point forward-difference formula)}$$

  $$f'(0.6) \approx \frac{f(0.7) - f(0.5)}{2 \cdot 0.1} \qquad \text{(midpoint forward-difference formula)}$$

  $$f'(0.6) \approx \frac{f(0.5) - f(0.6)}{-0.1} \qquad \text{(two-point backward-difference formula)}$$

- Possible aproximations of $f'(0.7)$:

  $$f'(0.7) \approx \frac{f(0.6) - f(0.7)}{-0.1} \qquad \text{(two-point backward-difference formula)}$$

  $$f'(0.7) \approx \frac{-3f(0.7) + 4f(0.6) - \frac{1}{2}f(0.5)}{2 \cdot (-0.1)} \qquad \text{(three-point backward-difference formula)}$$

  No forward-difference formulas can be used to approximate $f'(0.7)$.

## Richardson's extrapolation - Example 1

The Taylor series of $1/(1 - x^2)$ about $x = 0$ (convergent for $|x| < 1$) is

$$1 + x^2 + x^4 + x^6 + x^8 + O(x^{10})$$

Use Richardson's extrapolation to approximate 1 with $h = 1/10$ and truncation error $O(h^6)$.

Since

$$\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + +x^8 + O(x^{10}),$$

we have

$$1 = \frac{1}{1-h^2} - h^2 - h^4 - h^6 - h^8 - O(h^{10}).$$

Therefore

$N_1(h) = \dfrac{1}{1-h^2}$ and $N_2(h) = \frac{1}{3}\left[4N_1(h/2) - N_1(h)\right] = \dfrac{-5h^2+4}{h^4-5h^2+4}$.

Thus, $N_1(1/10) = \frac{100}{99} \approx 1.010101010$ with $|1 - N_1(1/10)| \approx 0.01010101010$

and $N_2(1/10) = \frac{39500}{39501} \approx 0.9999746842$ with $|1 - N_2(1/10)| \approx 0.00002531581479$

### Richardson's extrapolation - Problem 8 of Section 4.2

The forward-difference formula can be expressed as

$$f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

Use Richardson's extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.

*Solution*:

The formula

$$f'(x_0) = \frac{1}{12h}\left[f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)\right]$$

is $O(h^3)$.

### Richardson's extrapolation - Example 2

Use Richardson's extrapolation to midpoint formula to obtain $O(h^4)$ formula for $f'(x_0)$.

*Solution*:

With

$$N_1(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right],$$

we have,

$$N_2(h) = 2N_1(h/2) - N_1(h) = \frac{2}{h}f\left(x_0 + \frac{h}{2}\right) - \frac{2}{h}f\left(x_0 - \frac{h}{2}\right) + \frac{1}{2h}f(x_0-h) - \frac{1}{2h}f(x_0+h).$$

After changing $h/2$ to $h$, we obtain,

$$\boxed{N_2(h) = \frac{1}{h}\left[f(x_0+h) - f(x_0-h) + \frac{1}{4}f(x_0-2h) - \frac{1}{4}f(x_0+2h)\right].}$$

**Note:** *The resulting formula is consistent since as one can easily check,*

$$\lim_{h\to 0}\frac{1}{h}\left[f(x_0+h) - f(x_0-h) + \frac{1}{4}f(x_0-2h) - \frac{1}{4}f(x_0+2h)\right] = f'(x_0).$$

We also have

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \frac{h^5}{5!}f^{(5)}(x_0) + \frac{h^6}{6!}f^{(6)}(x_0) + \frac{h^7}{7!}f^{(7)}(x_0) + O(h^8)$$

and

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) - \frac{h^5}{5!}f^{(5)}(x_0) + \frac{h^6}{6!}f^{(6)}(x_0) - \frac{h^7}{7!}f^{(7)}(x_0) + O(h^8).$$

Next,

$$f(x_0+h) - f(x_0-h) = 2hf'(x_0) + 2\frac{h^3}{3!}f^{(3)}(x_0) + 2\frac{h^5}{5!}f^{(5)}(x_0) + 2\frac{h^7}{7!}f^{(7)}(x_0) + O(h^9),$$

and

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] + K_1h^2 + K_2h^4 + K_3h^6 + \dots.$$

The last expansion shows that after using Richardson's extrapolation, the truncation error of $N_2(h)$ is $\boxed{O(h^4)}$.

**Round-off error instability**

We have

$$f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} = f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} + \frac{f(x_0+h) - \tilde{f}(x_0+h) - (f(x_0) - \tilde{f}(x_0))}{h},$$

and thus,

$$f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} = -\frac{h}{2}f''(\xi) + \frac{e(x_0+h) - e(x_0)}{h}.$$

If $|f''(x)|$ is bounded by $M$ and the errors by $\epsilon$, we have

$$\boxed{\left|f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h}\right| \le \frac{h}{2}M + \frac{2\epsilon}{h}.}$$

The derivative of $E(h) = \frac{h}{2}M + \frac{2\epsilon}{h}$ is $E'(h) = \frac{M}{2} - \frac{2\epsilon}{h^2}$. Next, $E'(h) = 0$ when $h = 2\sqrt{\epsilon/M}$. Since $E''(h) = 4\epsilon/h^3 > 0$ for $h > 0$, the function $E(h)$ has the absolute minimum at $h = 2\sqrt{\epsilon/M}$. Therefore, $\boxed{h = 2\sqrt{\epsilon/M}}$ is the optimal value of $h$.

For $f(x) = \frac{1}{2x}$, $f''(x) = 1/x^3$ and with $x_0 = 1$, $M = \max_{x\ge 1} f''(x) = 1$. Therefore, for $\epsilon = 10^{-6}$, the optimal $h = 2\cdot 10^{-3}$, and

$$\boxed{f'(1) \approx \frac{\frac{1}{2(1+2\cdot 10^{-3})} - \frac{1}{2}}{2\cdot 10^{-3}} = -\frac{250}{501} \approx -0.499002.}$$

**Note:** *Since the exact value of $f'(1) = -1/2$, the actual error is*

$$\left|\frac{1}{2} - \frac{250}{501}\right| = \frac{1}{1002},$$

*while the error bound from the formula*

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(\xi), \quad h > 0, \quad x_0 \le \xi \le x_0 + h,$$

*with optimal $h = 2\cdot 10^{-3}$, is $10^{-3}$ and, of course, we have $\frac{1}{1002} < 10^{-3}$.*

## Problem 24 of Section 4.1

Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$

## Solution

We have Taylor expansions:

$$\begin{aligned}
f(x_0-h) &= f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(x_0) - \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5);\\
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{1}{2}h^2 f''(x_0) + \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5);\\
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \frac{4}{3}h^3 f'''(x_0) + \frac{2}{3}h^4 f^{(4)}(x_0) + O(h^5);\\
f(x_0+3h) &= f(x_0) + 3hf'(x_0) + \frac{9}{2}h^2 f''(x_0) + \frac{9}{2}h^3 f'''(x_0) + \frac{27}{8}h^4 f^{(4)}(x_0) + O(h^5);
\end{aligned}$$

Since the five-point formula uses $f(x_0-h)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$, expand $Af(x_0-h) + Bf(x_0+h) + Cf(x_0+2h) + Df(x_0+3h)$ in Taylor series using the above expansions:

$$\begin{aligned}
&Af(x_0-h) + Bf(x_0+h) + Cf(x_0+2h) + Df(x_0+3h)\\
&= f(x_0)(A+B+C+D) + f'(x_0)h\left[-A+B+2C+3D\right] + f''(x_0)h^2\left(\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D\right)\\
&+ f'''(x_0)h^3\left(-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D\right) + f^{(4)}(x_0)h^4\left(\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D\right) + O(h^5)
\end{aligned}$$

We want to eliminate the terms containing $f''(x_0)$, $f'''(x_0)$, and $f^{(4)}(x_0)$ and have the coefficient of $f'(x_0)$ equal 1. In order to accomplish that we need to solve the following system of linear equations:

$$\begin{aligned}
-A + B + 2C + 3D &= 1;\\
\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D &= 0;\\
-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D &= 0;\\
\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D &= 0.
\end{aligned}$$

The solution is $A = -\frac{1}{4}$, $B = \frac{3}{2}$, $C = -\frac{1}{2}$, and $D = \frac{1}{12}$. Thus

$$-\frac{1}{4}f(x_0-h) + \frac{3}{2}f(x_0+h) - \frac{1}{2}f(x_0+2h) + \frac{1}{12}f(x_0+3h) = f(x_0)\left(-\frac{1}{4} + \frac{3}{2} - \frac{1}{2} + \frac{1}{12}\right) + hf'(x_0) + O(h^5)$$

Solving for $f'(x_0)$ gives

$$f'(x_0) = -\frac{1}{h}\left[\frac{10}{12}f(x_0) + \frac{1}{4}f(x_0-h) - \frac{3}{2}f(x_0+h) + \frac{1}{2}f(x_0+2h) - \frac{1}{12}f(x_0+3h)\right] + O(h^5),$$

which finally can be written as

$$f'(x_0) = \frac{1}{12h}\left[-f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)\right] + O(h^5).$$

## Additional Practice Problem

Show that

$$\lim_{h\to 0}\frac{1}{12h}\left[-f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)\right] = f'(x_0).$$