Separable differential equations

Math. 481a, Spring 2026

A first order differential equation

$$\frac{dy}{dx} = f(x, y),$$

when $f(x, y) = g(x)h(y)$ for some functions g and h.

Example 1.

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1 \tag{1}$$

The differential equation (1) can be written as

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$

Integrating the left side with respect to y and the right side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{2}$$

where c is arbitrary constant. To determine the solution satisfying the initial condition $y(0) = -1$ we substitute $x = 0$ and $y = -1$ into (2), obtaining $c = 3$. Hence the solution is given **implicitly** by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{3}$$

In this case we can obtain the solution explicitly by solving (3) for y in terms of x. Since (3) is quadratic in y we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{4}$$

Equation (4) gives two solutions of the differential equation, only one of which , however, satisfies the given initial condition $y(0) = -1$. This is the solution corresponding to the minus sign in (4), so the solution of (1) is

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{5}$$

Remark 1. *Note that if the plus sign is chosen by mistake in (4), then we obtain the solution of the same differential equation that satisfies the initial conditions $y(0) = 3$.*

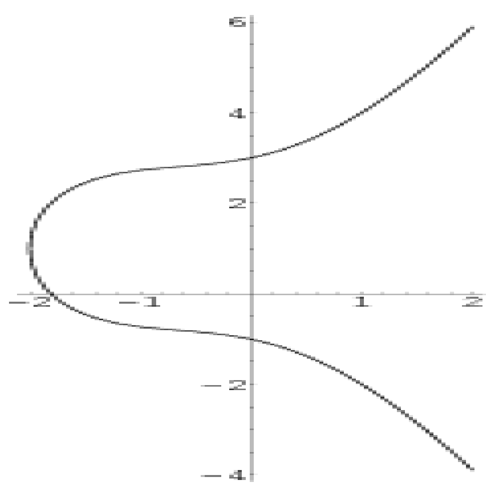

FIGURE 1. The integral curve of $y' = (3x^2 + 4x + 2)/2(y-1)$ that satisfies the initial condition $y(0) = -1$.

Please observe that the graph in Figure 1 does not represent a function !!!! Which branch of the graph represents the solution satisfying the initial condition $\mathbf{y(0) = -1}$ *and which branch of the graph represents the solution with the initial condition* $\mathbf{y(0) = 3}$ *?*

Finally, we can determine the interval in which solution (5) is valid. Indeed, since $x^3 + 2x^2 + 2x + 4 = (x+2)(x^2+2)$ has only real zero at $x = -2$, the function $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$ is defined for $x > -2$. The solutions of the initial value problem (1) and some other integral (dashed) curves of the differential equation (1) are shown in Figure 2.

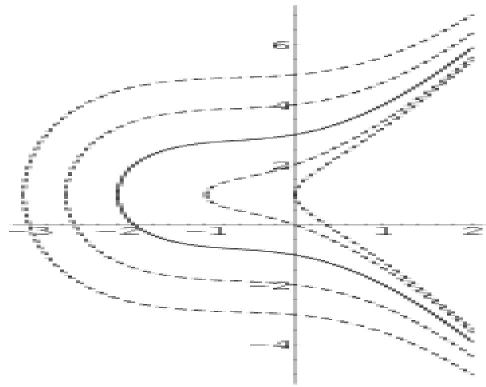

FIGURE 2. Integral curves of $y' = (3x^2+4x+2)/2(y-1)$.

Example 2.
Solve the initial value problem

$$\frac{dy}{dx} = y^2\cos(\pi x), \quad y(0) = -\frac{1}{2} \tag{6}$$

The equation (6) can be written as

$$\frac{dy}{y^2} = \cos(\pi x)dx.$$

Integrating the left side with respect to y and the right side with respect to x gives

$$-\frac{1}{y} = \frac{1}{\pi}\sin(\pi x) + c, \quad \text{or} \quad y(x) = -\frac{\pi}{\sin(\pi x) + c\pi}.$$

To determine c, set $x = 0$ and solve:

$$y(0) = -\frac{1}{c} = -\frac{1}{2} \quad \Longrightarrow \quad c = 2.$$

Thus the solution to the initial value problem (6) is

$$y(x) = -\frac{\pi}{\sin(\pi x) + 2\pi}.$$

For the graph of the solution of (6) see Figure 3, below.

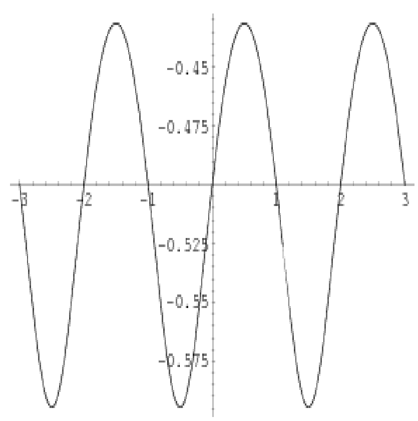

FIGURE 3. Graph of the solution to (6).

Example 3.
Solve the differential equation

$$\frac{dy}{dx} = \frac{y^2}{1+x^2+y^2+x^2y^2}$$

From $\left(\frac{1}{y^2}+1\right)dy = \frac{1}{1+x^2}dx$ we obtain $-\frac{1}{y} + y = \tan^{-1}(x) + c$, or $y = \frac{1}{(1-c)-\tan^{-1}(x)}$, or $y = \frac{1}{\bar{c}-\tan^{-1}(x)}$.

Example 4.

Solve the differential equation

$$\frac{dN}{dt} + N = Nt\exp(t+2).$$

From $\frac{1}{N}dN = (t\exp(t+2) - 1)$ we obtain

$$\begin{aligned} \log|N| &= t\exp(t+2) - \exp(t+2) - t + c, \quad \text{or} \\ N &= \pm\exp(c)\exp\left[t\exp(t+2) - \exp(t+2) - t\right], \quad \text{or} \\ N &= \bar{c}\exp\left[t\exp(t+2) - \exp(t+2) - t\right]. \end{aligned}$$

Example 5.

Solve the differential equation

$$exp(x)\frac{dy}{dx} = 2x.$$

From $dy = 2x\exp(-x)dx$ we obtain $y = -2x\exp(-x) - 2\exp(-x) + c$.

Example 6.

Solve the differential equation

$$y\exp(x)\frac{dy}{dx} = \exp(-y) + \exp(-2x - y)$$

From $y\exp(y)dy = \left[\exp(-x) + \exp(-3x)\right]dx$ we obtain $y\exp(y) - \exp(y) + \exp(-x) + \dfrac{1}{3}\exp(-3x) = c$.

This implicit solution cannot be solve for y as a function of x.