Notes for section 5.4

Math. 481a, Spring 2026

Runge-Kutta Methods

$\boxed{a_1 f(t+\alpha_1, y+\beta_1) \text{ form}}$

The goal is to find a_1, α_1, and β_1 with the property that

$$a_1 f(t+\alpha_1, y+\beta_1) \approx T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$$

with error not greater than $O(h^2)$. We have

$$f'(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot y'(t) \quad \text{and} \quad y'(t) = f(t,y).$$

Thus

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y) \cdot f(t,y) \tag{1}$$

Next, we expand $f(t+\alpha_1, y+\beta_1)$ in Taylor polynomial of degree one about (t,y):

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+a_1, y+\beta_1), \tag{2}$$

where

$$R_1(t+a_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t\,\partial y}(\xi,\mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\mu),$$

for some ξ between t and $t+\alpha_1$ and μ between y and $y+\beta_1$.

Now, we compare coefficients of f and its derivatives in (1) and (2)

$$f(t,y): \quad a_1 = 1; \qquad\qquad \frac{\partial f}{\partial t}(t,y): \quad a_1\alpha_1 = \frac{h}{2};$$

and

$$\frac{\partial f}{\partial y}(t,y): \quad a_1\beta_1 = \frac{h}{2} f(t,y).$$

The parameters a_1, α_1, and β_1 are uniquely determined by

$$a_1 = 1, \qquad \alpha_1 = \frac{h}{2}, \qquad \beta_1 = \frac{h}{2} f(t,y).$$

thus,

$$T^{(2)}(t,y) = f\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) - R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right),$$

where

$$R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \frac{h^2}{4}\frac{\partial^2 f}{\partial t\,\partial y}(\xi,\mu) + \frac{h^2}{8}\frac{\partial^2 f}{\partial y^2}(\xi,\mu).$$

When all the second-order derivatives partial derivatives of f are bounded, $R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right)$ is $O(h^2)$.

The method resulting from replacing $T^{(2)}(t,y)$ in Taylor method of order two by $f\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right)$ is a Runge-Kutta method known as the *Midpoint method*.

Midpoint Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf\left(t_i+\frac{h}{2}, w_i+\frac{h}{2}f(t_i,w_i)\right), \quad \text{for each } i = 0,1,2,\ldots,N-1. \end{aligned}$$

$$\boxed{a_1 f(t,y) + a_2 f(t+\alpha_2, y+\delta_2 f(t,y)) \text{ form}}$$

The appropriate method for approximating

$$T^{(3)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y) + \frac{h^2}{6} f''(t,y)$$

seems to be four-parameter form $a_1 f(t,y) + a_2 f(t+\alpha_2, y+\delta_2 f(t,y)))$. However, there is insufficient flexibility to match the term

$$\frac{h^2}{6}\left[\frac{\partial f}{\partial y}(t,y)\right]^2 f(t,y).$$

An interesting $O(h^2)$ (instead of $O(h^3)$ methods can be derived.

With $a_1 = a_2 = \dfrac{1}{2}$ and $\alpha_2 = \delta_2 = h$ we obtain,

Modified Euler Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_i + h, w_i + hf(t_i, w_i))\right], \quad \text{for each } i = 0, 1, 2, \ldots, N-1. \end{aligned}$$

Difference method of order 3

$$\boxed{a_1 f(t,y) + a_2 f(t+\alpha_1, y+\delta_1 f(t+\alpha_2, y+\delta_2 f(t,y))) \text{ form}}$$

It can be shown that $T^{(3)}(t,y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$a_1 f(t,y) + a_2 f(t+\alpha_1, y+\delta_1 f(t+\alpha_2, y+\delta_2 f(t,y))).$$

With $a_1 = \dfrac{1}{4}$, $a_2 = \dfrac{3}{4}$, $\alpha_1 = \delta_1 = \dfrac{2}{3}h$, and $\alpha_2 = \beta_2 = \dfrac{1}{3}h$, we obtain,

Heun's Method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \frac{h}{4}\left\{ f(t_i, w_i) + 3f\left[t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf\left(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i, w_i)\right)\right]\right\}, \text{ for each } i = 0, 1, 2, \ldots, N-1. \end{aligned}$$

In practice, however, this third order Runge-Kutta method is not generally used. The most common method is the following Runge-Kutta method of order four:

Runge-Kutta Order Four

$$\begin{aligned} w_0 &= \alpha, \\ k_1 &= hf(t_i, w_i), \\ k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}k_1\right), \\ k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}k_2\right), \\ k_4 &= hf(t_{i+1}, w_i + k_3), \\ w_{i+1} &= w_i = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \quad \text{for each } i = 0, 1, 2, \ldots, N-1. \end{aligned}$$

Another derivation of the Midpoint Method

Since Euler's method is of order one an extrapolation method can be applied to obtain a method of order two. So assume that two integrations have been performed with the step size $h_1 = h$ and the second with the step size $h_2 = h/2$ by means of the Euler method, up to a given abscissa t. The resulting values w_n and w_{2n} after n and $2n$ steps, respectively, satisfy approximately

$$w_n \approx y(t) + c_1 h + O(h^2),$$
$$w_{2n} \approx y(t) + c_1 \frac{1}{2} h + O(h^2).$$

By applying Richardson's extrapolation, the extrapolated quantity

$$\tilde{w} = 2w_{2n} - w_n \approx y(t) + O(h^2).$$

Instead of integrating a differential equation by means of Euler's method with two different step sizes, it is better to directly apply the extrapolation to the values that result from an integration step with the step size h and from a double step with half the step size. In both cases we start from the approximate point (t_k, w_k). An ordinary step size h yields the value

$$w_{k+1}^{(1)} = w_k + hf(t_k, w_k).$$

A double step with the step size $h/2$ produces

$$w_{k+\frac{1}{2}}^{(2)} = w_k + \frac{h}{2} f(t_k, w_k)$$
$$w_{k+1}^{(2)} = w_{k+\frac{1}{2}}^{(2)} + \frac{h}{2} f\left(t_k + \frac{h}{2}, w_{k+\frac{1}{2}}^{(2)}\right).$$

The Richardson's extrapolation applied to $w_{k+1}^{(2)}$ and $w_{k+1}^{(1)}$ defines the extrapolated approximation

$$\begin{aligned} w_{k+1} = 2w_{k+1}^{(2)} - w_{k+1}^{(1)} &= 2w_{k+\frac{1}{2}}^{(2)} + hf\left(t_k + \frac{h}{2}, w_{k+\frac{1}{2}}^{(2)}\right) - w_k - hf(t_k, w_k) \\ &= 2w_k + hf(t_k, w_k) + hf\left(t_k + \frac{h}{2}, w_{k+\frac{1}{2}}^{(2)}\right) - w_k - hf(t_k, w_k) \\ &= w_k + hf\left(t_k + \frac{h}{2}, w_k + \frac{h}{2} f(t_k, w_k)\right). \end{aligned}$$

But this is the **Midpoint Method**.

Use of a quadrature formula to derive a new difference method

The differential equation $y' = f(t, y)$ considered on the interval $[t_i, t_{i+1}]$ (with $h = t_{i+1} - t_i$) is equivalent to

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(s, y(s))\, ds. \tag{3}$$

If we apply the Simpson's quadrature formula to the right hand side of the equation (3) with step size equal to $h/2$, we obtain

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(s, y(s))\, ds \approx \frac{h}{6}\left[f\left(t_i, y(t_i)\right) + 4f\left(t_i + \frac{h}{2}, y\left(t_i + \frac{h}{2}\right)\right) + f\left(t_i + h, y(t_i + h)\right)\right].$$

The corresponding implicit finite difference formula is

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{6}\left[f\left(t_i, w_i\right) + 4f\left(t_i + \frac{h}{2}, w_{i+\frac{1}{2}}\right) + f\left(t_i + h, w_{i+1}\right)\right]. \end{aligned} \tag{4}$$

Finally, applying Euler's Method to $w_{i+\frac{1}{2}}$ and w_{i+1} appearing in the right hand of (4), we obtain the following finite difference method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{6}\left[f\left(t_i, w_i\right) + 4f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) + f\left(t_i + h, w_i + hf(t_i, w_i)\right)\right]. \end{aligned}$$