Notes for section 5.1

Math. 481a, Spring 2026

Separable equations

Find the solution of the initial value problem

$$y' = (1-2t)y^2, \qquad y(0) = -\frac{1}{6} \tag{1}$$

and determine the interval in which the solution is defined.

Equation (1) is separable; integrating (when $y \neq 0$) we obtain

$$\int \frac{dy}{y^2} = \int (1-2t)\,dt, \quad \text{or} \quad -\frac{1}{y} = t - t^2 + c, \quad \text{or} \quad y = \frac{1}{t^2 - t - c}, \quad y(0) = -\frac{1}{6} \quad \Longrightarrow \quad c = 6,$$

and

$$y(t) = \frac{1}{t^2 - t - 6}.$$

Remark 1: The graph of $1/(t^2-t-6)$ *consists of three branches, but only the middle one represents the solution to* (1)*. (See Figure 1.) However, it is easy to see that the left branch can be a solution too, for example, when the initial value is given by* $y(-4) = 1/14$*. The same is true for the right branch, for example, when the initial value is* $y(4) = 1/10$.

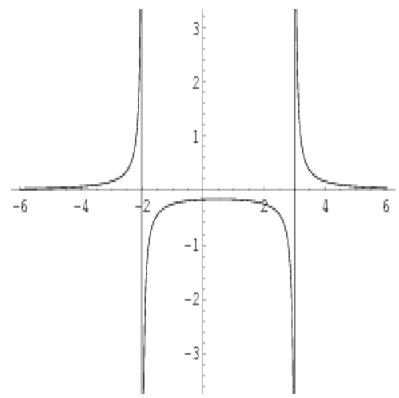

FIGURE 1. The graph of the function $1/(t^2 - t - 6)$

Solution is defined in the interval containing 0 where $y = 1/(t^2 - t - 6)$ makes sense, i.e., $-2 < t < 3$. Note that $y(-2)$ and $y(3)$ are not defined.

Solve the initial value problem

$$y' = \frac{3t^2}{3y^2 - 4}, \qquad y(1) = 0, \tag{2}$$

and determine the interval in which the solution is valid.

Equation (2) is separable; integrating (when $3y^2 - 4 \neq 0$) we obtain

$$\int (3y^2 - 4)\,dy = \int 3t^2\,dt \quad \Longrightarrow \quad y^3 - 4y = t^3 + c \quad \Longrightarrow \quad \text{and} \quad y(1) = 0 \quad \Longrightarrow \quad c = -1,$$

and thus

$$y^3 - 4y - t^3 + 1 = 0.$$

We look for a solution defined in the interval containing 1. The integral curve $y^3 - 4y - t^3 + 1 = 0$ has vertical tangent lines at the points where y' is unbounded, i.e., when $3y^2 - 4 = 0$ (remember, $y' = 3t^2/(3y^2-4)$), or when $y = \pm 2/\sqrt{3}$. This gives us $t^3 - 1 = \pm\left\{\left(2/\sqrt{3}\right)^3 - 4\left(2/\sqrt{3}\right)\right\} = \pm 16/(3\sqrt{3})$. In other words, the sought interval is either the set of t for which $|t^3 - 1| < 16/(3\sqrt{3})$ or the set of t for which $|t^3 - 1| > 16/(3\sqrt{3})$. Only the first choice guarantees that $t = 1$ belongs to it. (**Do you know why ?**) Thus, the interval is given by $|t^3 - 1| < 16/(3\sqrt{3})$, or by

Do you know which part of the graph in Figure 2 represents the solution to the initial value problem (2) *?*

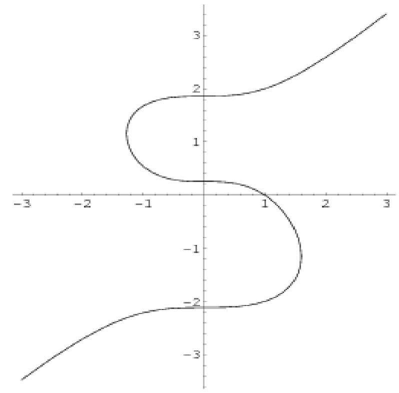

FIGURE 2. The integral curve of (2) satisfying the initial condition $y(1) = 0$.

$$-1.28 \approx \left(1 - \frac{16}{3\sqrt{3}}\right)^{1/3} < t < \left(1 + \frac{16}{3\sqrt{3}}\right)^{1/3} \approx 1.60.$$

Linear equations

Solve the following differential equation:

$$ty' + (1+t)y = \exp(-t)\sin(4t), \qquad y(\pi/8) = 0.$$

An integrating factor is

$$\exp\left(\int \left[1 + \frac{1}{t}\right] dt\right) = t\exp(t),$$

so that

$$\frac{d}{dt}[t\exp(t)y] = \sin(4t) \quad \text{or} \quad t\exp(t)y = -\frac{1}{4}\cos(4t) + c.$$

Thus,

$$t\exp(t)y = -\frac{1}{4}\cos(4t) + c, \quad \text{or} \quad y = \frac{c}{t}\exp(-t) - \frac{1}{4t}\exp(-t)\cos(4t).$$

From the initial condition $y(\pi/8) = 0$ we obtain $c = 0$ and the solution is

$$y(t) = -\frac{1}{4t}\exp(-t)\cos(4t).$$

Solve the following initial value problem:

$$xy' + y = \exp(x), \qquad y(1) = 2.$$

The integrating factor is x, thus the solution of $(xy)' = \exp(x)$ is

$$y(x) = \frac{1}{x}\exp(x) + \frac{C}{x}.$$

The initial condition $y(1) = 2$ yield $C = 2 - e$ and the solution is

$$y(x) = \frac{1}{x}\exp x + \frac{2-e}{x}.$$