Notes for section 4.3

Math. 481a, Spring 2026

Numerical Integration

A **numerical quadrature** is an approximation of $\int_a^b f(x)\,dx$ of the form

$$\sum_{i=0}^{n} a_i f(x_i), \qquad \text{for suitable choices of } a_i \text{ and } x_0, x_1, \ldots, x_n \in [a,b],\ i = 0, 1, \ldots, n.$$

The Lagrange interpolating polynomials will be used to derive various numerical quadratures. We select distinct nodes $x_0, x_1 \ldots, x_n \in [a,b]$ and integrate Lagrange interpolating polynomial

$$P_n(x) = \sum_{j=0}^{n} f(x_i) L_{n,i}(x).$$

We obtain

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))\,dx, \tag{1}$$

where $\xi(x) \in [a,b]$ for each x and

$$a_i = \int_a^b L_{n,i}\,dx, \qquad \text{for each } i = 0, 1, \ldots, n.$$

The quadrature formula is

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i),$$

with error given by

$$E(f) = \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))\,dx,$$

Special cases: Trapezoidal rule

Let $x_0 = a,\ x_1 = b,\ h = b - a$, and $n = 1$.

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

Formula (1) with $n = 1$ becomes

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] dx + \frac{1}{2} \int_{x_0}^{x_1} (x - x_0)(x - x_1) f''(\xi(x))\,dx \tag{2}$$

Since $(x - x_0)(x - x_1)$ does not change sign for $x \in [x_0, x_1]$, the Weighted Mean Value Theorem for Integrals shows that for some $\xi_1 \in (x_0, x_1)$

$$\int_{x_0}^{x_1} (x - x_0)(x - x_1) f''(\xi(x))\,dx = f''(\xi_1) \int_{x_0}^{x_1} (x - x_0)(x - x_1)\,dx$$

The integral on the right hand side of the last equation becomes

$$\int_{x_0}^{x_1} (x-x_0)(x-x_1)\,dx = \left[\frac{x^3}{3} - \frac{x_1+x_0}{2}x^2 + x_0x_1x\right]\Bigg|_{x_0}^{x_1} = -\frac{h^3}{6}.$$

Thus, (2) becomes

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] - \frac{h^3}{12}f''(\xi_1). \qquad \textbf{(Trapezoidal Rule)} \tag{3}$$

Note 1

The error term for the Trapezoidal rule (3) *involves f''; hence the rule gives the exact result when applied to any function whose second derivative is identically zero, i.e., any polynomial of degree one or less.*

Special cases: Simpson's rule

Let $x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h$. The third Taylor polynomial about x_1 is

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 + \frac{f^{(3)}(x_1)}{6}(x-x_1)^3 + \frac{f^{(4)}(\xi(x))}{24}(x-x_1)^4;$$

integrating both sides of the last formula between x_0 and x_1 we obtain

$$\int_{x_0}^{x_2} f(x)\,dx = \left[f(x_1)(x-x_0) + \frac{f'(x_1)}{2}(x-x_1)^2 + \frac{f''(x_1)}{6}(x-x_1)^3 + \frac{f^{(3)}(x_1)}{24}(x-x_1)^4 + \right]\Bigg|_{x_0}^{x_2} \tag{4}$$

$$+\frac{1}{24}\int_{x_0}^{x_2} f^{(4)}(\xi(x))(x-x_1)^4\,dx$$

As before $(x-x_1)^4$ has the same sign for $x \in [x_0, x_2]$ and the Weighted Mean Value Theorem for Integrals implies that

$$\frac{1}{24}\int_{x_0}^{x_1} f^{(4)}(\xi(x))(x-x_1)^4\,dx = \frac{f^{(4)}(\xi_1)}{24}\int_{x_0}^{x_2}(x-x_1)^4\,dx = \frac{f^{(4)}(\xi_1)}{120}(x-x_1)^5\Bigg|_{x_0}^{x_2} = \frac{f^{(4)}(\xi_1)}{120}2h^5 = \frac{f^{(4)}(\xi_1)}{60}h^5.$$

The formula (4) becomes

$$\int_{x_0}^{x_2} f(x)\,dx = 2hf(x_1) + \frac{h^3}{3}f''(x_1) + \frac{f^{(4)}(\xi_1)}{60}h^5. \tag{5}$$

Now, approximating $f''(x_1)$ by (see section 1 of Chapter 4)

$$f''(x_1) = \frac{1}{h^2}\left[f(x_0) - 2f(x_1) + f(x_2)\right] - \frac{h^2}{12}f^{(4)}(\xi_2)$$

the formula (5) has the form

$$\int_{x_0}^{x_2} f(x)\,dx = 2hf(x_1) + \frac{h^3}{3}\left\{\frac{1}{h^2}\left[f(x_0)-2f(x_1)+f(x_2)\right] - \frac{h^2}{12}f^{(4)}(\xi_2)\right\} + \frac{f^{(4)}(\xi_1)}{60}h^5$$

$$= \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] - \frac{h^5}{12}\left[\frac{1}{3}f^{(4)}(\xi_2) - \frac{1}{5}f^{(4)}(\xi_1)\right]$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi) \qquad \textbf{(Simpson's Rule)}$$

Note 2

The error term for the Simpson's involves $f^{(4)}$; hence the rule gives the exact result when applied to any function whose fourth derivative is identically zero, i.e., any polynomial of degree three or less.

Definition. The **degree of accuracy**, or **precision**, of a quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, \ldots, n$

Closed Newton-Cotes formulas

If **n is even**, $f \in C^{n+2}[a, b]$, $h = (b - a)/n$, $x_0 = a$, and $x_i = x_0 + ih$ $(i = 0, 1, \ldots, n)$, then there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\cdots(t-n)\,dt.$$

If **n is odd**, $f \in C^{n+1}[a, b]$, $h = (b - a)/n$, $x_0 = a$, and $x_i = x_0 + ih$ $(i = 0, 1, \ldots, n)$, then there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n)\,dt.$$

Case n=1; Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \frac{h^3}{12} f''(\xi), \qquad x_0 < \xi < x_1.$$

Case n=2; Simpson's rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi), \qquad x_0 < \xi < x_2.$$

Case n=3; Simpson's Three-Eighths rule

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi), \qquad x_0 < \xi < x_3.$$

Case n=4;

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right] - \frac{8h^7}{945} f^{(6)}(\xi), \qquad x_0 < \xi < x_4.$$

Open Newton-Cotes formulas

If **n is even**, $f \in C^{n+2}[a,b]$, $h = (b-a)/(n+2)$, $x_{-1} = a$, $x_0 = a + h$, $x_i = x_0 + ih$ $(i = 1, \ldots, n, n+1)$, and $x_{n+1} = b$, then there exists $\xi \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2(t-1)\cdots(t-n)\,dt.$$

If **n is odd**, $f \in C^{n+1}[a,b]$, $h = (b-a)/(n+2)$, $x_{-1} = a$, $x_0 = a + h$, $x_i = x_0 + ih$ $(i = 1, \ldots, n, n+1)$, and $x_{n+1} = b$, then there exists $\xi \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{n+1} t(t-1)\cdots(t-n)\,dt.$$

n=0; Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi), \qquad x_{-1} < \xi < x_1.$$

n=1;

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1)\right] + \frac{3h^3}{4} f''(\xi), \qquad x_{-1} < \xi < x_2.$$

n=2;

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] + \frac{14h^5}{45} f^{(4)}(\xi), \qquad x_{-1} < \xi < x_3.$$

n=3;

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}\left[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right] + \frac{95h^5}{144} f^{(4)}(\xi), \qquad x_{-1} < \xi < x_4.$$