Notes for section 4.2

Math. 481a, Spring 2026

Richardson's extrapolation

Suppose we approximate an unknown value M by $N(h)$ and for each $h > 0$ the truncation error involved with the approximation has the form

$$M - N(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots ,$$

where K_1, K_2, K_3, … are unknown constants. Since the truncation error is $O(h)$ we expect $M - N(h) \approx K_1 h$. We assume here that there is not a large variation in magnitudes among the constants K_1, K_2, K_3, … .

The goal of extrapolation is to produce formulas that approximate M with higher-order truncation error than $O(h)$. We start with

$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots . \tag{1}$$

The above formula is valid for all $h > 0$, in particular, it is valid if h is replaced by $h/2$:

$$M = N\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots . \tag{2}$$

Subtracting equation (1) from twice equation (2) eliminates K_1 and gives

$$M = \left[2N\left(\frac{h}{2}\right) - N(h)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots .$$

With the notation $N_1(h) = N(h)$ and

$$N_2(h) = \left[2N_1\left(\frac{h}{2}\right) - N_1(h)\right] = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right],$$

we have the $O(h^2)$ approximation formula for M:

$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \cdots . \tag{3}$$

If we now replace h by $h/2$ in (3), we obtain,

$$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8}h^2 - \frac{3K_3}{32}h^3 - \cdots . \tag{4}$$

Subtracting equation (3) from 4 times equation (4) and dividing both sides by 3 gives an $O(h^3)$ formula for approximating M,

$$M = \left[N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3}\right] + \frac{K_3}{8}h^3 + \cdots = N_3(h) + \frac{K_3}{8}h^3 + \cdots \tag{5}$$

The process can be continued to construct an $O(h^4)$ approximation of M,

$$N_4(h) = N_3\left(\frac{h}{2}\right) + \frac{N_3(h/2) - N_3(h)}{7},$$

an $O(h^5)$ approximation of M,

$$N_5(h) = N_4\left(\frac{h}{2}\right) + \frac{N_4(h/2) - N_4(h)}{15},$$

and so on. In general, if M can be written in the form

$$M = N(h) + \sum_{j=1}^{m-1} K_j h^j + O(h^m), \tag{6}$$

for each $j = 2, 3, \dots, m$, we have an $O(h^j)$ approximation of M:

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{2^{j-1} - 1}, \tag{7}$$

Extrapolation can be also applied whenever the truncation error has the form

$$\sum_{j=1}^{m-1} K_j h^{\alpha_j} + O\left(h^{\alpha_m}\right), \tag{8}$$

with $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m$.

Example 1.
Consider the Taylor polynomial of f about x_0:

$$\begin{aligned} f(x) = f(x_0) &+ f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{6}f'''(x_0)(x - x_0)^3 + \frac{1}{24}f^{(4)}(x_0)(x - x_0)^4 \\ &+ \frac{1}{120}f^{(5)}(x_0)(x - x_0)^5 + \frac{1}{720}f^{(6)}(x_0)(x - x_0)^6 + \frac{1}{5040}f^{(7)}(\xi)(x - x_0)^7, \end{aligned}$$

for some ξ between x and x_0. Evaluating the above formula at $x_0 + h$ and $x_0 - h$ we obtain

$$\begin{aligned} f(x_0 + h) = f(x_0) &+ f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(x_0)h^5 \\ &+ \frac{1}{720}f^{(6)}(x_0)h^6 + \frac{1}{5040}f^{(7)}(\xi_1)h^7 \end{aligned} \tag{9}$$

and

$$\begin{aligned} f(x_0 - h) = f(x_0) &- f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(x_0)h^5 \\ &+ \frac{1}{720}f^{(6)}(x_0)h^6 - \frac{1}{5040}f^{(7)}(\xi_2)h^7. \end{aligned} \tag{10}$$

Subtracting equation (10) from equation (9) and dividing both sides of the resulting equation by $2h$, we get

$$\frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] = f'(x_0) + \frac{1}{6}f'''(x_0)h^2 + \frac{1}{120}f^{(5)}(x_0)h^4 + \frac{h^6}{5040}\left[f^{(7)}(\xi_1) + f^{(7)}(\xi_2)\right]. \tag{11}$$

Solving equation (11) for $f'(x_0)$, we obtain

$$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] - \frac{1}{6}f'''(x_0)h^2 - \frac{1}{120}f^{(5)}(x_0)h^4 - \frac{h^6}{5040}\left[f^{(7)}(\xi_1) + f^{(7)}(\xi_2)\right],$$

or equivalently (after applying the Intermediate Value Theorem to $f^{(7)}(x)$)

$$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] - \frac{1}{6}f'''(x_0)h^2 - \frac{1}{120}f^{(5)}(x_0)h^4 - \frac{h^6}{2520}f^{(7)}(\xi). \tag{12}$$

The last formula shows that with

$$N_1(h) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right],$$

the approximation of $f'(x_0)$ has the truncation error as in (8) with $\alpha_j = 2j$.

Example 2 (Problem 1c, page 189)
Start with the centered difference formula (see formula (12))

$$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] - \frac{h^2}{6}f^{(3)}(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

Apply Richardson's extrapolation method to determine $N_3(h)$, $N_4(h)$, $N_5(h)$, and $N_6(h)$, approximations to $f'(1.05)$ for $f(x) = 2^x \sin(x)$ and $h = 0.4$.

We have,

$$\begin{aligned}
N_1(h) &= \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right];\\
N_1(0.4) &= \frac{1}{2\cdot 0.4}\left[f(1.05+0.4) - f(1.05-0.4)\right] \approx 1.957245799;\\
N_2(h) &= N_1\left(\frac{h}{2}\right) + \frac{N_1(h/2) - N_1(h)}{3};\\
N_2(0.4) &= N_1(0.2) + \frac{N_1(0.2) - N_1(0.4)}{3} \approx 2.275261094;\\
N_3(h) &= N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{15};\\
N_3(0.4) &= N_2(0.2) + \frac{N_2(0.2) - N_2(0.4)}{15} \approx 2.275145948;\\
N_4(h) &= N_3\left(\frac{h}{2}\right) + \frac{N_3(h/2) - N_3(h)}{63};\\
N_4(0.4) &= N_3(0.2) + \frac{N_3(0.2) - N_3(0.4)}{63} \approx 2.275145831;\\
N_5(h) &= N_4\left(\frac{h}{2}\right) + \frac{N_4(h/2) - N_4(h)}{255};\\
N_5(0.4) &= N_4(0.2) + \frac{N_4(0.2) - N_4(0.4)}{255} \approx 2.275145831;\\
N_6(h) &= N_5\left(\frac{h}{2}\right) + \frac{N_5(h/2) - N_5(h)}{1023};\\
N_6(0.4) &= N_5(0.2) + \frac{N_5(0.2) - N_5(0.4)}{1023} \approx 2.275145846.
\end{aligned}$$

The derivative of f is $f'(x) = (\ln 2)2^x \sin(x) + 2^x \cos(x)$, and $f'(1.05) \approx 2.275145842$.

Derivation of the five-point method from three-point method

The same arguments used in Example 1 show that

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{1}{6}f'''(x_0)h^2 - \frac{1}{120}f^{(5)}(\xi_1)h^4, \tag{13}$$

for some ξ between $x_0 - h$ and $x_0 + h$.

By replacing h with $2h$ in (13), we obtain

$$f'(x_0) = \frac{1}{4h}\left[f(x_0+2h) - f(x_0-2h)\right] - \frac{4}{6}h^2 f^{(3)}(x_0) - \frac{16}{120}h^4 f^{(5)}(\xi_2). \tag{14}$$

Multiplying equation (13)by 4 and subtracting equation (14) produces

$$3f'(x_0) = \frac{2}{h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{1}{4h}\left[f(x_0+2h) - f(x_0-2h)\right] - \frac{h^4}{30}f^{(5)}(\xi_1) + \frac{2h^4}{15}f^{(5)}(\xi_2)$$

Using the Intermediate Value Theorem to $f^{(5)}$ (if it is continuous) we can replace $f^{(5)}(\xi_1)$ and $f^{(5)}(\xi_2)$ by a common value $f^{(5)}(\xi)$ and thus obtaining the following five-point formula

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right] + \frac{h^4}{30}f^{(5)}(\xi).$$