# Notes for section 3.2 and 3.3

## Math. 481a, Spring 2026

## Divided Differences

Suppose that $P_n(x)$ is the $n$th Lagrange polynomial that agrees with the function $f$ at the distinct numbers $x_0, x_1, \ldots, x_n$. It has the form:

$$P_n(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \tag{1}$$

where

$$L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}, \quad k = 0, 1, \ldots n.$$

This polynomial is unique.

**Remark 1**
There is another way to construct the polynomials in (1). Indeed, we seek a polynomial $Q_n(x)$,

$$Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k, \tag{2}$$

such that $Q_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots n$. By considering the coefficients $\alpha_k$ in (2) as unknowns, we have the system of $n+1$ linear equations

$$Q_n(x_i) = \alpha_0 + \alpha_1 x_i + \cdots + \alpha_n x_i^n = f(x_i), \qquad i = 0, 1, \ldots n \tag{3}$$

This system has a unique solution, if the determinant of the coefficient matrix is nonzero

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix} = \prod_{i<j}(x_i - x_j) \equiv \prod_{j=0}^{n-1}\left[\prod_{i=j+1}^{n}(x_i - x_j)\right]$$

This determinant is often called a *Vandermonde* determinant. Since the $\{x_i\}$ are distinct points, this determinant is nonzero and the system (3) may be uniquely solved for $\alpha_i$ to determine the interpolating polynomial.

In addition to (1), $P_n(x)$ can be represented in different forms. Another form is introduced so that the addition of a further support point is simplified as in the Neville's algorithm. For example, we can express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}), \tag{4}$$

for yet to be determined coefficients $a_0, a_1, \ldots, a_n$.

The unknown coefficients $a_0, a_1, \ldots, a_n$ can be successively derived from the set of interpolation conditions:

$$\begin{aligned}
P_n(x_0) &= a_0 && = f(x_0)\\
P_n(x_1) &= a_0 + a_1(x_1-x_0) && = f(x_1)\\
P_n(x_2) &= a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) && = f(x_2)\\
P_n(x_3) &= a_0 + a_1(x_3-x_0) + a_2(x_3-x_0)(x_3-x_1) + a_3(x_3-x_0)(x_3-x_1)(x_3-x_0) && = f(x_3)
\end{aligned}$$

etc

Form (4) is also used in divided differences of $f$. It is easy to find $a_0$ in (4). Indeed, we have

$$a_0 = P_n(x_0) = f(x_0).$$

Next, for $a_1$, we have

$$P_n(x_1) = f(x_1) = a_0 + a_1(x_1-x_0) = f(x_0) + a_1(x_1-x_0).$$

Thus,

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Now, we define divided-difference notation. The *zeroth divided difference* of the function $f$ with respect to $x_i$ is

$$f[x_i] = f(x_i).$$

The remaining divided-differences are defined inductively. The *first divided difference* of the function $f$ with respect to $x_i$ and $x_{i+1}$ is defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \tag{5}$$

The *second divided difference* of the function $f$ with respect to $x_i$, $x_{i+1}$, and $x_{i+2}$ is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} = \frac{\dfrac{f(x_{i+2}) - f(x_{i+1})}{x_{i+2} - x_{i+1}} - \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}{x_{i+2} - x_i}$$

Similarly, after the $(k-1)$st divided differences

$$f[x_i, x_{i+1}, x_{i+2}, \dots x_{i+k-1}] \quad \text{and} \quad f[x_{i+1}, x_{i+2}, \dots x_{i+k-1}, x_{i+k}]$$

have been determined, the *kth divided difference* relative to $x_i, x_{i+1}, x_{i+2}, \dots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \dots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots x_{i+k-1}, x_{i+k}] - f[x_i, x_{i+1}, x_{i+2}, \dots x_{i+k-1}]}{x_{i+k} - x_i}$$

The process ends with the single *nth divided difference*,

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

We have seen above that $a_1 = f[x_0, x_1]$ and $a_2 = f[x_0, x_1, x_2]$. In general,

$$a_k = f[x_0, x_1, \dots, x_k], \qquad k = 0, 1, \dots, n.$$

So $P_n(x)$ can be expressed as

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0.x_1, \dots, x_k](x - x_0)(x - x_1)\dots(x - x_{k-1}) \tag{6}$$

**Remark 2**

The value of $f[x_0.x_1, \dots, x_k]$ does **NOT** depend on the order of the numbers $x_0, x_1, \dots x_k$.

The Mean Value Theorem applied to (5) (when $i = 1$) implies that when $f'$ exists there exists $\xi$ between $x_0$ and $x_1$ such that

$$f[x_0, x_1] = f'(\xi).$$

We have

**Theorem.** Suppose $f \in C^n[a, b]$, and $x_0, x_1, \dots, x_n \in [a, b]$ are distinct numbers. Then there exists $\xi \in (a, b)$ (generally unknown) such that

$$f[x_0, x_1 \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

*Proof.* Let $g(x) = f(x) - P_n(x)$. We have $f(x_i) = P_n(x_i)$, for $i = 0, 1, \dots, n$. Also, $g$ has $n+1$ distinct zeros in $[a, b]$. The generalized Rolle's Theorem implies that there exists $\xi \in (a, b)$ such that $g^{(n)}(\xi) = 0$. Therefore, we have

$$P_n^{(n)}(\xi) = f^{(n)}(\xi).$$

Since $P_n(x)$ is a polynomial of degree $n$ whose leading coefficient is $f[x_0, x_1, \dots, x_n]$, we have

$$P_n^{(n)}(x) = n! f[x_0, x_1, \dots, x_n], \quad \text{for } x \in [a, b].$$

Therefore,

$$f[x_0, x_1 \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

□

## Newton Forward-Difference and Newton Backward-Difference Formulas

Next we arrange $x_0, x_1, \ldots, x_n$ with equal spacing. Define $h = x_{i+1} - x_i$ for each $i = 0, 1, \ldots, n-1$, and let $x = x_0 + sh$. Then the differences $x - x_i = (s-i)h$ and (6) reads

$$\begin{aligned} P_n(x) &= P_n(x_0+sh) = f[x_0] + shf[x_0,x_1] + s(s-1)h^2 f[x_0,x_1,x_2] + \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0,x_1,\ldots,x_n] \\ &= f[x_0] + \sum_{k=1}^{n} s(s-1)\ldots(s-k+1)h^k f[x_0.x_1,\ldots,x_k] = f[x_0] + \sum_{k=1}^{n} \binom{s}{k}(k!)h^k f[x_0.x_1,\ldots,x_k]. \end{aligned}$$

Next using forward difference notation

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0,x_1,x_2] = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{1}{2h}\left(\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right) = \frac{1}{2h^2}\Delta^2 f(x_0),$$

and, in general,

$$f[x_0,x_1\ldots,x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Finally, (6) becomes

$$P_n(x) = f(x_0) + \sum_{k=1}^{n}\binom{s}{k}\Delta^k f(x_0). \qquad \textbf{Newton Forward-Difference Formula} \tag{7}$$

Sometimes, we may want to reorder interpolating points (nodes) as $x_n, x_{n-1}, \ldots, x_1.x_0$. With equal spacing $x = x_n + sh$ and $x = x_i + (s+n-i)h$, the interpolatory formula ((6)) becomes

$$\begin{aligned} P_n(x) &= P_n(x_n+sh) \\ &= f[x_n] + shf[x_n,x_{n-1}] + s(s+1)h^2 f[x_n,x_{n-1},x_{n-2}] + \cdots + s(s+1)\ldots(s+n-1)h^n f[x_n,x_{n-1},\ldots,x_1,x_0]. \end{aligned} \tag{8}$$

The backward difference for a sequence $\{p_n\}$ is defined by

$$\nabla p_n = p_n - p_{n-1}, \qquad \text{for } n \geq 1,$$

and recursively for higher powers

$$\nabla^k p_n = \nabla(\nabla^{k-1}p_n), \qquad \text{for } k \geq 2.$$

Applying it to divided differences we get

$$f[x_n,x_{n-1}] = \frac{1}{h}\nabla f(x_n), \qquad f[x_n,x_{n-1},x_{n-2}] = \frac{1}{2h^2}\nabla^2 f(x_n),$$

and, in general,

$$f[x_n,x_{n-1},\ldots,x_{n-k}] = \frac{1}{k!h^k}\nabla^k f(x_n).$$

Equation (8) becomes

$$P_n(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n) + \cdots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f(x_n) \tag{9}$$

Using the extension of binomial notation to real values of $s$

$$\binom{-s}{k} = \frac{-s(-s-1)\ldots(-s-k+1)}{k!} = (-1)^k\frac{s(s+1)\ldots(s+k-1)}{k!},$$

(9) reads

$$P_n(x) = f(x_n) + \sum_{k=1}^{n}(-1)^k\binom{-s}{k}\nabla^k f(x_n). \qquad \textbf{Newton Backward-Difference Formula} \tag{10}$$