Notes for section 2.4

Math. 481a, Spring 2026

Definition. Suppose $p_n \to p$ as $n \to \infty$ with $p_n \neq p$ for all n. If there exist $\lambda > 0$ and $\alpha > 0$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda, \tag{1}$$

then $\{p_n\}$ converges to p of order α, with asymptotic error constant λ.

An iterative scheme $p_n = g(p_{n-1})$ is said to be of order α if $\{p_n\}$ converges to the solution $p = g(p)$ of order α.

Two cases of convergence are important:

- If $\alpha = 1$ (and $0 < \lambda \leq 1$) the sequence is **linearly convergent**.
- If $\alpha = 2$ the sequence is **quadratically convergent**.

Example 1, (Problem 6, page 85)

For $p_n = 1/n$ we have

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

and, for $\alpha > 1$,

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n^\alpha}} = \infty,$$

so $p_n \to 0$ linearly.

For $p_n = 1/n^2$ we have

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$

and $p_n \to 0$ linearly.

Example 2. For $p_n = 1/2^n$ we have

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2},$$

so $p_n \to 0$ linearly.

Example 3, (Problem 7, page 85) For $p_n = 1/n^k$ (with k positive integer) we have

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$

so $p_n \to 0$ linearly.

Example 4, (Problem 8, page 85)

For $p_n = 10^{-2^n}$ we have

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

so $p_n \to 0$ quadratically.

The sequence $p_n = 10^{-n^k}$ does not converge to zero quadratically for any $k > 1$. Indeed,

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n\to\infty} 10^{2n^k - (n+1)^k} = \infty.$$

Example 5, (Problem 9b, page 85)

The sequence $p_n = 10^{-\alpha^n}$, with $\alpha > 0$ converges to 0 of order α. Indeed,

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha^{n+1}}} = 1$$

Theorem 1. Assume $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that $g'(x)$ is continuous on (a,b) and there exists $0 < k < 1$ with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b). \tag{2}$$

If $g'(p) \neq 0$, then for any $p_0 \in [a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only linearly to the unique fixed point $p \in [a,b]$.

Proof. From the Fixed Point Theorem $p_n \to p$, as $n \to \infty$. For any fixed $n \geq 1$, the Mean Value Theorem implies

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p),$$

where ξ_n is between p_n and p. Since $p_n \to p$, as $n \to \infty$, $\xi_n \to p$. Thus,

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n\to\infty} g'(\xi_n) = g'(p), \quad \text{since } g' \text{ is continuous on } (a,b).$$

Therefore

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|.$$

and, if $|g'(p)| \neq 0$, fixed-point iteration converges linearly with asymptotic error constant $\lambda = |g'(p)|$. □

Theorem 2. Assume p solves $g(x) = x$. Suppose that $g'(p) = 0$ and $g''(x)$ is continuous on an open interval I containing p with $|g''(x)| \leq M$ for $x \in I$. Then there exists $\delta > 0$ such that for $p_0 \in [p-\delta, p+\delta]$, the fixed-point iteration $p_n = g(p_{n-1})$ $(n \geq 1)$ converges at least quadratically to p. Moreover, for large enough n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Proof. Choose $0 < k < 1$ and $\delta > 0$ such that $[p-\delta, p+\delta] \subseteq I$ and $|g'(x)| \leq k$ on $[p-\delta, p+\delta]$.

The same arguments as in the proof of Theorem 3 of the Notes for section 2.2 show that the sequence $p_n = g(p_{n-1})$ is contained in $[p-\delta, p+\delta]$ and $\lim_{n\to\infty} p_n = p$.

Now, we expand $g(x)$ in Taylor series about $x = p$:

$$g(x) = g(p) + g'(p)\cdot(x-p) + \frac{1}{2}\cdot g''(\xi)\cdot(x-p)^2, \quad \text{where } \xi \text{ is between } x \text{ and } p.$$

The last equality together with $g(p) = p$ and $g'(p) = 0$ implies

$$g(x) = p + \frac{1}{2}\cdot g''(\xi)\cdot(x-p)^2.$$

Substituting $x = p_n$ gives

$$p_{n+1} = g(p_n) = p + \frac{1}{2}\cdot g''(\xi_n)\cdot(p_n - p)^2, \quad \text{where } \xi_n \text{ is between } p_n \text{ and } p$$

or equivalently,

$$p_{n+1} - p = \frac{1}{2}\cdot g''(\xi_n)\cdot(p_n - p)^2.$$

Since $p_n \to p$, as $n \to \infty$, $\xi_n \to p$, Thus,

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2}. \quad \text{since } g'' \text{ is continuous on } [p-\delta, p+\delta].$$

This implies that p_n converges at least quadratically to p, with asymptotic error constant $\lambda = |g''(p)|/2$.

Since $|g''(x)| < M$ on $[p-\delta, p+\delta]$, we also have for sufficiently large n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

□