Notes for section 2.2

Math. 481a, Spring 2026

Definition. A number p is a **fixed point** of a given function g if $g(p) = p$.

Root-finding and fixed-point problems are equivalent in the following sense:

- If p is such that $f(p) = 0$ the p is a fixed point for $g(x) = x + f(x)$. There are infinite many such functions. Indeed every function $g_i(x) = x + if(x)$, for $i = 1, 2, \ldots$. has the property that $g_i(p) = p$.
- Conversely, if g has a fixed point p then the function defined by $f(x) = x - g(x)$ has a zero at p.

Example 1
The function $g(x) = x^2$ defined on $\mathbb{R}$ has two fixed points: $x = 0$ and $x = 1$.

Theorem 1. Assume $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Then g has a fixed point in $[a, b]$.
If, in addition, $g'(x)$ exists on (a, b) and there exists $0 < k < 1$ with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a, b), \tag{1}$$

then the fixed point is unique.

Proof. There are two possible cases.
Case 1 $g(a) = a$ or $g(b) = b$.
The function g has a fixed point and we are done.

Case 2 $g(a) \neq a$ and $g(b) \neq b$.
By the assumption, $g(x) \in [a, b]$ for $x \in [a, b]$, therefore $g(a) > a$ and $g(b) < b$.

The function h defined on $[a, b]$ defined by

$$h(x) = g(x) - x$$

is continuous on $[a, b]$ with the properties

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

By the Intermediate Value Theorem, there exists $p \in [a, b]$ such that $h(p) = 0$. Equivalently, $g(p) = h(p) + p = p$.

Assume, in addition, that (1) is satisfied. We proceed by contradiction. Suppose that $p \neq q$ are fixed points of g in $[a, b]$. Without losing any generality we can assume that $p > q$.

The Mean Value Theorem implies that there exists $\xi \in (q, p)$ such that

$$g(p) - g(q) = g'(\xi) \cdot (p - q).$$

The last equation implies that

$$|p - q| = |g(p) - g(q)| = |g'(\xi)| \cdot |p - q| \leq k \cdot |p - q| < |p - q|.$$

Since $p \neq q$, we have reached a contradiction. □

Example 2
It is easy to check by inspection that the function g defined on the interval $[3, 4]$ by $g(x) = (x^2 - 1)/3$ has a unique fixed point $p = (3 + \sqrt{13})/2$. However, $g(4) = 5$ and $g'(x) = (2x)/3 > 1$ on $[3, 4]$, so g does **not** satisfy the assumptions of Theorem 1. This example shows that the hypotheses of Theorem 1 are sufficient to guarantee a unique fixed point but are not necessary.

Fixed-point iteration

Choose p_0 in the domain of g and define the sequence $\{p_n\}$ by

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1.$$

If g is continuous and the sequence $p_n \to p$ as $n \to \infty$, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

and p is a fixed point of g. This technique is called **fixed-point iteration** or **functional iteration**.

Theorem 2 (Fixed-Point Theorem)**.** Assume $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that $g'(x)$ exists on (a,b) and there exists $0 < k < 1$ with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b). \tag{2}$$

Then for any $p_0 \in [a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges to the unique fixed-point $p \in [a,b]$.

Proof. Theorem 1 implies that a unique point p exists in $[a,b]$. Also, since $g : [a,b] \to [a,b]$, the sequence $\{p_n\}$ is defined for all $n \geq 0$ and $p_n \in [a,b]$ for all n. Applying the Mean Value Theorem for each n, we have

$$|p_n - p| = |g(p) - g(p_{n-1})| = |g'(\xi_n)| \cdot |p_{n-1} - p| \leq k|p_{n-1} - p|,$$

where ξ_n is between p and p_n and thus belongs to the interval (a,b). Applying the last inequality inductively, we obtain

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|. \tag{3}$$

However, $0 < k < 1$, and thus $k^n \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = 0,$$

and the sequence $\{p_n\}$ converges to the unique fixed point p of g. □

Corollary. Under the assumptions of Theorem 2, the error involved in using p_n to approximate p is given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|,$$

for all $n \geq 1$.

Proof. The first inequality follows the inequality (3)

$$|p_n - p| \leq k^n|p_0 - p| \leq k^n \max\{p_0 - a, b - p_0\}.$$

For the second bound we notice that as in the proof of Theorem 2

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq k|p_n - p_{n-1}| \leq \cdots \leq k^n|p_1 - p_0|,$$

and thus, for $m > n \geq 1$

$$\begin{aligned} |p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots + p_{n+1} - p_n| \leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-1}| + \cdots + |p_{n+1} - p_n| \\ &\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n|p_1 - p_0| = k^n \left(\sum_{i=0}^{m-n-1} k^i\right)|p_1 - p_0|. \end{aligned}$$

Since $0 < k < 1$, the geometric series in the parenthesis converges as $m \to \infty$ and its sum is bounded by

$$\sum_{i=0}^{m-n-1} k^i \leq \sum_{i=0}^{\infty} k^i = \frac{1}{1-k}.$$

Therefore, for any $n \geq 1$,

$$|p - p_n| = \lim_{m\to\infty} |p_m - p_n| \leq k^n \lim_{m\to\infty} \left(\sum_{i=0}^{m-n-1} k^i \right) |p_1 - p_0| \leq \frac{k^n}{1-k} |p_1 - p_0|.$$

$\square$

Example 3. (Problem 14d, page 64)
For the equation

$$x - \cos x = 0$$

determine the interval on which fixed point iteration converges. Estimate the number of iterations necessary to obtain approximation accurate to within 10^{-5}.

Using $g(x) = \cos x$ on the interval $[0, 1]$ we have $g : [0, 1] \to [0, 1]$ and $|g'(x)| = |-\sin x| = \sin x$ is increasing on the interval $[0, 1]$. This also means that $0 \leq |g'(x)| = \sin x \leq \sin 1 = k \approx 0.8414709848$. With $p_0 = 0$, we have,

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} = k^n < 10^{-5},$$

or equivalently,

$$n > \frac{\ln(0.00001)}{\ln k} \approx 66.70148074.$$

However, our tolerance is met with $p_{30} = 0.73908230$. The approximation of the fixed point with 10-digit accuracy is 0.7390851332.

Example 4. (Problem 20, page 65)
For $A > 0$ let $g(x) = 2x - Ax^2$.
(a) Show that if fixed-point iteration converges to a nonzero limit then the limit is $p = 1/A$.
(b) Find an interval about $1/A$ for which fixed point iteration converges, provided p_0 is in this interval.

$$\text{(a)} \quad p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} [2p_{n-1} - A(p_{n-1})^2] = 2p - Ap^2.$$

Solving for p gives $p = 1/A$.

(b) Consider the interval $\left[\frac{1}{2A}, \frac{3}{2A}\right]$. It contains $1/A$. Next,
$g'(x) = 2 - 2Ax = 0$ only when $x = 1/A$. $g(x)$ is continuous and $g'(x)$ exists for all x. Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A},$$

the Extreme Value Theorem yields

$$\frac{3}{4A} \leq g(x) \leq \frac{1}{A}, \quad \text{for } x \in \left[\frac{1}{2A}, \frac{3}{2A}\right].$$

Therefore,

$$g : \left[\frac{1}{2A}, \frac{3}{2A}\right] \to \left[\frac{1}{2A}, \frac{3}{2A}\right].$$

Also, for $x \in \left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left| x - \frac{1}{A} \right| < \frac{1}{2A},$$

so

$$|g'(x)| = 2A \left| x - \frac{1}{A} \right| < 2A \left(\frac{1}{2A} \right) = 1.$$

Thus, the assumptions of the Fixed-Point Theorem 2 are satisfied for $g(x) = 2x - Ax^2$ on the interval $\left[\frac{1}{2A}, \frac{3}{2A}\right]$. Fixed-point iteration converges to $p = 1/A$ for any $p_0 \in \left[\frac{1}{2A}, \frac{3}{2A}\right]$.

Additional remarks

Remark 1.
If g' is continuous at p, the two assumptions of Theorem 2:

(1) $g : [a, b] \to [a, b]$ and
(2) $|g'(x)| \le k < 1, \quad$ for all $x \in (a, b)$,

can be relaxed. The following result is true:

Theorem 3 (Problem 26, page 65)**.** Let $g \in C[a, b]$ and g' exists on (a, b). Assume that g' is continuous at a fixed point p. If $|g'(p)| < 1$, then there exists $\delta > 0$ such that for p_0 satisfying $|p_0 - p| < \delta$, the fixed point iteration $p_n = g(p_{n-1})$ converges to p as $n \to \infty$.

Proof. Let $\epsilon = (1 - |g'(p)|)/2$. Since g' is continuous at p there exists $\delta > 0$ such that for $x \in [p - \delta, p + \delta]$, we have

$$|g'(x) - g'(p)| < \epsilon.$$

Thus,

$$|g'(x)| < |g'(p)| + \epsilon = k < 1, \quad \text{for } x \in [p - \delta, p + \delta]. \tag{4}$$

By the Mean Value Theorem, we have for $x \in [p - \delta, p + \delta]$,

$$|g(x) - g(p)| = |g'(c)| \cdot ||x - p| \le k|x - p| < |x - p|, \quad \text{where } c \text{ is between } x \text{ and } p.$$

The last inequality implies that $g : [p - \delta, p + \delta] \to [p - \delta, p + \delta]$. Furthermore, by (4), $|g'(x)| \le k < 1$ on the interval $[p - \delta, p + \delta]$. Therefore the assumptions of the Fixed-Point Theorem 2 are satisfied for g defined on the interval $[p - \delta, p + \delta]$. Theorem 2 implies that the fixed point iteration $p_n = g(p_{n-1})$ $(n = 1, 2, \dots)$, with $p_0 \in [p - \delta, p + \delta]$, converges to p as $n \to \infty$. □

Remark 2.
The assumption that $|g'(p)| < 1$ is needed for the functional iterates to converge to the fixed point p. Indeed, we have the following result:

Theorem 4 (Problem 19, page 65)**.** Let $g \in C[a, b]$ and g' exists on (a, b). Assume that g' is continuous at a fixed point p of g. If $|g'(p)| > 1$, then there exists $\delta > 0$ such that if $|p_0 - p| < \delta$, then

$$|p_1 - p| > |p_0 - p|.$$

Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is further away, so the fixed-point iteration does not converge if $p_0 \ne p$.

Proof. Let $\epsilon = |g'(p)| - 1 > 0$. Since g' is continuous at p there exists $\delta > 0$ such that whenever $0 < |x - p| < \delta$ we have

$$|g'(x) - g'(p)| < \epsilon = |g'(p)| - 1.$$

Hence, for any x satisfying $0 < |x - p| < \delta$, we also have

$$|g'(x)| = |g'(p) - g'(p) + g'(x)| \ge |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is such that $0 < |p_0 - p| < \delta$, the Mean Value Theorem implies that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)| \cdot |p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ and

$$|p_1 - p_0| = |g'(\xi)| \cdot |p_0 - p| > |p_0 - p|.$$

□