Notes for section 2.1

Math. 481a, Spring 2026

Theorem 1 (Intermediate Value Theorem)**.** Assume $f \in C[a,b]$. For any w between $f(a)$ and $f(b)$, there exists at least one $c \in [a,b]$ such that $f(c) = w$.

Example 1
For $f(x) = x^3 - x - 1$, the Intermediate Value Theorem implies that the exists $c \in [1,2]$ such that $f(c) = c^3 - c - 1 = 0$.
Indeed, f as a polynomial is a continuous function on $[a,b]$ for any $a < b$. We have $f(1) = -1$ and $f(2) = 5$. Thus, $0 \in [-1,5]$ and there exists at least one $c \in [1,2]$ with $f(c) = 0$.

Theorem 2 (Convergence of the Bisection method)**.** Assume $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_1^\infty$ approximating a zero p of f with

$$|p_n - p| \le \frac{b-a}{2^n}, \qquad \text{for } n \ge 1. \tag{1}$$

Proof. For each $n \ge 1$ we have

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2^2}(b_{n-2} - a_{n-2}) = \cdots = \frac{1}{2^{n-1}}(b_1 - a_1) = \frac{1}{2^{n-1}}(b-a). \tag{2}$$

Now, for all $n \ge 1$ $p_n = \dfrac{1}{2}(a_n + b_n)$ and $p \in (a_n, b_n)$. This implies that

$$|p_n - p| \le \frac{1}{2}(b_n - a_n). \qquad \textit{(Do you know why?)}$$

Therefore, using (2), we have

$$|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2^n}(b-a).$$

The last inequality implies that

$$\lim_{n\to\infty} p_n = p$$

and

$$p_n = p + O\left(\frac{1}{2^n}\right).$$

$\square$

Note: *It is interesting to observe that the bound in (1) depends only on a, b, and on required accuracy (from which n is determined). This bound does* ***not*** *depend on f.*

Example
Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4$ lying in the interval $[1,4]$. Find an approximation to the root with this degree of accuracy.

The function $f(x) = x^3 + x - 4$ is a polynomial and therefore it is a continuous function of x on any interval $[a,b]$. In our case $a = 1$ and $b = 4$. Also $f(1) = -2 < 0$ and $f(4) = 64 > 0$. By Theorem 1 (Intermediate Value Theorem), f has at least one zero in the interval $[1,4]$.

If p is such zero and p_n is its approximation from the Bisection method, then from (1) we have

$$|p_n - p| \le \frac{4-1}{2^n} = \frac{3}{2^n} \le 10^{-3}.$$

Equivalently,

$$3000 \le 2^n \quad \Longleftrightarrow \quad \ln(3000) \le n\ln(2) \quad \Longleftrightarrow \quad n \ge \frac{\ln(3000)}{\ln(2)} \approx 11.55074679.$$

Thus, the smallest number of iterations required to achieve the desired accuracy of 10^{-3} is $n = 12$.

The corresponding approximation to the root is $p_{12} = 1.378662110$.