Notes for section 1.3

Math. 481a, Spring 2026

The algorithms with the property that small changes in the initial data produce correspondingly small changes in the final results are called **stable**.

Some algorithms are stable only for certain choices of initial data. Such algorithms are called **conditionally stable**.

Suppose an error with magnitude E_0 is introduced at some stage in the calculations and after n subsequent operations the error is E_n. How fast E_n grows is critical:

Definition 1. Suppose that E_0 denotes an initial error and E_n is the error after n subsequent operations. If

$$E_n \approx CnE_0, \qquad \text{where } C \text{ is a constant independent of } n,$$

then the growth of error is called **linear**.

If

$$E_n \approx C^n E_0, \qquad \text{for some } C > 1,$$

then the growth of error is called **exponential**.

Linear growth error is unavoidable; and when C and E_0 small the final results are quite acceptable. On the other hand, exponential growth should be avoided, regardless of the size of E_0.

If

$$E_n \approx Cn^p E_0, \qquad \text{for some } C \text{ independent of } n \text{ and } p > 1,$$

the error growth is called **polynomial**. Such growth is faster than linear but slower than exponential growth.

Ilustration, pages 32-34.

It is easy to verify by inspection that the recurrence equation

$$p_n = \frac{10}{3}p_{n-1} - p_{n-2}, \qquad n = 2, \dots,$$

has the solution

$$p_n = c_1\left(\frac{1}{3}\right)^n + c_2 3^n,$$

for any $c_1, c_2 \in \mathbb{R}$. For the initial conditions

$$p_0 = 1 \quad \text{and} \quad p_1 = \frac{1}{3},$$

$c_1 = 1$ and $c_2 = 0$, and the solution is

$$p_n = \left(\frac{1}{3}\right)^n, \quad n = 1, 2, \dots.$$

Now, assume that five-digit rounding is used to compute the terms of the sequence $\{p_n\}$. Thus, $\hat{p}_0 = 1.0000$ and $\hat{p}_1 = 0.33333$. The corresponding constants c_1 and c_2 have the values

$$\hat{c}_1 = 1.0000 \quad \text{and} \quad \hat{c}_2 = -0.12500 \times 10^{-5},$$

while the sequence $\hat{p}_n$ is

$$\hat{p}_n = 1.0000\left(\frac{1}{3}\right)^n - \left(0.12500 \times 10^{-5}\right) \cdot 3^n.$$

The round-off error is

$$p_n - \hat{p}_n = \left(0.12500 \times 10^{-5}\right) \cdot 3^n.$$

It grows exponentially. This results in large inaccuracies. Indeed, for example, for $n = 8$,

the corrected $p_8 = 0.15242 \times 10^{-3}$, the computed $\hat{p}_8 = -0.92872 \times 10^{-2}$, and the relative error is 6×10^1.

On the other hand, the recursive equation

$$p_n = 2p_{n-1} - p_{n-2}, \qquad n = 2, 3, \dots$$

has the solution

$$p_n = c_1 + c_2 n.$$

for any $c_1, c_2 \in \mathbb{R}$. For the initial data $p_0 = 1$ and $p_1 = \frac{1}{3}$, we have $c_1 = 1$ and $c_2 = -\frac{2}{3}$, giving the solution:

$$p_n = 1 - \frac{2}{3}n.$$

As before, the five-digits rounding for p_0 and p_1 results in

$$\hat{c}_1 = 1.0000 \quad \text{and} \quad \hat{c}_2 = -0.66667,$$

and the corresponding solutions:

$$\hat{p}_n = 1.0000 - 0.66667n.$$

The round-off error is

$$p_n - \hat{p}_n = \left(0.66667 - \frac{2}{3}\right)n.$$

It grows linearly. The resulting inaccuracies are small. For example, for $n = 8$,

the corrected $p_8 = -0.43333 \times 10^1$, the computed $\hat{p}_8 = -0.43334 \times 10^1$, and the relative error is 2×10^{-5}.

Definition 2. Assume $\beta_n \to 0$ and $\alpha_n \to \alpha$ as $n \to \infty$. If a positive constant K exists with

$$|\alpha_n - \alpha| \leq K|\beta_n|, \qquad \text{for large } n,$$

then we say that the sequence $\{\alpha_n\}$ converges to α with **rate of convergence** $O(\beta_n)$. We often write $\alpha_n = \alpha + O(\beta_n)$.

In practical situations, we almost always use

$$\beta_n = \frac{1}{n^p}, \qquad \text{for some } p > 0.$$

and we are interested in the largest value of p with $\alpha_n = \alpha + O(1/n^p)$.

For the following two sequences:

$$\alpha_n = \frac{2n^2 - 10n + 100}{n^3} \qquad \text{and} \qquad \hat{\alpha}_n = \frac{3n^3 - 10000n^2 + 1}{n^6},$$

we have

$$\alpha_n = 0 + O\left(\frac{1}{n}\right) \qquad \text{and} \qquad \hat{\alpha}_n = 0 + O\left(\frac{1}{n^3}\right),$$

correspondingly.

The rate of convergence for functions is defined similarly:

Definition 3. Assume $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$. If there exists a positive K with

$$|F(h) - L| \leq K|G(h)|, \qquad \text{for sufficiently small } h,$$

then we write $F(h) = L + O(G(h))$. In practical situations we use $G(h) = h^p$, with $p > 0$.

Problem 7a, page 36

Find the rate of convergence of

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1. \tag{1}$$

The Taylor series of $\sin(x)$ about $x = 0$ is

$$\sin(x) = x - \frac{1}{6}[\cos\xi(x)]x^3, \quad \text{for } \xi(x) \text{ between } 0 \text{ and } x.$$

Thus,

$$\left|\frac{\sin(h)}{h} - 1\right| = \frac{1}{6}|\cos\xi(h)||h|^2 \leq \frac{1}{6}|h|^2$$

and the function in (1) has the rate of convergence

$$\frac{\sin(h)}{h} = 1 + O(h^2).$$

Problem 7c, page 36

Find the rate of convergence of

$$\lim_{h\to 0} \frac{\sin(h) - h\cos(h)}{h} = 0. \tag{2}$$

Using the Taylor series of $\sin(x)$ about $x = 0$ from part (a) together with the Taylor series of $\cos(x)$ about $x = 0$,

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}[\cos\hat{\xi}(x)]x^4, \quad \text{for } \hat{\xi}(x) \text{ between } 0 \text{ and } x,$$

gives

$$\frac{\sin(h) - h\cos(h)}{h} = \frac{1}{2}h^2 - \frac{1}{6}[\cos\xi(h)]h^2 - \frac{1}{24}[\cos\hat{\xi}(h)]h^4.$$

Therefore, the function in (2) has the rate of convergence

$$\frac{\sin(h) - h\cos(h)}{h} = 0 + O(h^2).$$

Problem 7d, page 36

Find the rate of convergence of

$$\lim_{h\to 0} \frac{1 - \exp(h)}{h} = -1. \tag{3}$$

The Taylor series of $\exp(x)$ about $x = 0$ is

$$\exp(x) = 1 + x + \frac{1}{2}[\exp(\bar{\xi}(x))]x^2, \quad \text{for } \bar{\xi}(x) \text{ between } 0 \text{ and } x.$$

Thus,

$$\frac{1 - \exp(h)}{h} + 1 = \frac{1 - 1 - h - \frac{1}{2}[\exp(\bar{\xi}(h))]h^2}{h} + 1 = -\frac{1}{2}[\exp(\bar{\xi}(h))]h.$$

The function in (3) has the rate of convergence

$$\frac{1 - \exp(h)}{h} = -1 + O(h).$$