Notes for Chapter 1.1

Math. 481a, Spring 2026

Calculus Review

- The limits of functions
- Intermediate Value Theorem
- The Derivative and its interpretations
- Extreme Value Theorem
- The Mean Value Theorem for derivatives
- The Riemann integral (the definite integral)
- The Mean Value Theorem for integrals
- Taylor's Theorem

Theorem (Taylor's Theorem)**.** Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$. and $x_0 \in [a,b]$. Then for every $x \in [a,b]$, there exists $\xi(x)$ between x_0 and x such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_0^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}\,.$$

Example

Find the second and third Taylor polynomial for $f(x) = \cos(x)$ *about* $x_0 = 0$*. Use this polynomials to approximate* $\cos(0.01)$*. Use the third Taylor polynomial and its remainder to approximate* $\int_0^{0.1} \cos(x)\,dx$*.*

The second Taylor polynomial

$$\cos(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi(x),$$

where $\xi(x)$ is between 0 and x. For $x = 0.01$, we have

$$\cos(0.01) = 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin \xi(0.01) = 0.99995 + 0.1(6) \times 10^{-6} \sin \xi(0.01).$$

We don't know the value of $\xi(0.01)$, however, since $|\sin z| \le 1$ for any z, we have the following estimation for the error we are making in approximating $\cos(0.01)$ by 0.99995 (the second Taylor polynomial):

$$|\cos(0.01) - 0.99995| = 0.1(6) \times 10^{-6} |\sin \xi(0.01)| \le 0.1(6) \times 10^{-6}.$$

The error bound is much larger than the actual error. Indeed, the value of $\cos(0.01) = 0.999950000416665$ with 15 digits accuracy, and thus

$$|\cos(0.01) - 0.99995| = 0.416665 \times 10^{-9}.$$

The larger error bound comes from the fact that we have used rough estimation $|\sin\xi(0.01)| \leq 1$. However, for $0 \leq \xi < 0.01$, $|\sin z| \leq |z|$, and $|\sin\xi(0.01)| \leq 0.01$. If we use this better estimation, the resulting error bound becomes:

$$|\cos(0.01) - 0.99995| = 0.1(6) \times 10^{-6} |\sin\xi(0.01)| \leq 0.1(6) \times 10^{-8}.$$

The third Taylor polynomial

Since $f'''(0) = 0$, we have

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cos\tilde{\xi}(x),$$

where $\tilde{\xi}(x)$ is between 0 and x.

We note that the approximation is the same, but now we have better error estimation. Indeed, using the estimation $|\cos\tilde{\xi}(x)| \leq 1$, we have

$$|\cos(0.01) - 0.99995| = \frac{1}{24}(0.01)^4 |\cos\tilde{\xi}(x)| < 0.42 \times 10^{-9}.$$

The example illustrates two main objectives of numerical analysis:

(1) *find an approximation to the solution of the given problem,*
(2) *determine a bound for the error of the approximation.*

The third Taylor polynomial gives:

$$\int_0^{0.1} \cos x\,dx = \int_0^{0.1} \left(1 - \frac{1}{2}x^2\right)dx + \frac{1}{24}\int_0^{0.1} x^4 \cos\tilde{\xi}(x)\,dx = 0.1 - \frac{1}{6}(0.1)^3 + \frac{1}{24}\int_0^{0.1} x^4 \cos\tilde{\xi}(x)\,dx$$

with

$$\int_0^{0.1} \cos x\,dx \approx 0.1 - \frac{1}{6}(0.1)^3 = 0.0998(3).$$

Since $|\cos z| \leq 1$ for all z, the error bound is given by

$$\frac{1}{24}\left|\int_0^{0.1} x^4 \cos\tilde{\xi}(x)\,dx\right| \leq \frac{1}{24}\int_0^{0.1} x^4 |\cos\tilde{\xi}(x)|\,dx \leq \frac{1}{24}\int_0^{0.1} x^4\,dx = 0.8(3) \times 10^{-7}.$$

The actual error is smaller:

$$\left|\int_0^{0.1} \cos x\,dx - \left[0.1 - \frac{1}{6}(0.1)^3\right]\right| = 0.833134949 \times 10^{-7},$$

where 15 digits accuracy was used in evaluating the integral

$$\int_0^{0.1} \cos x\,dx = 0.0998334166468282.$$