

First order linear differential equations

Math. 481a, Spring 2026

A linear first order differential equation has the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

when P and f are given functions.

Example 1.

Find the general solution of

$$\frac{dy}{dx} + 2xy = 0.$$

Here $P(x) = 2x$ and $y(x) = c \exp\left(\int -2x dx\right) = c \exp(-x^2)$.

Example 2.

Find the solution of the initial value problem

$$\frac{dy}{dx} + (\sin x)y = 0, \quad y(0) = \frac{3}{2}.$$

Here $P(x) = \sin x$ so that

$$y(x) = \frac{3}{2} \exp\left(-\int_0^x \sin t dt\right) = \frac{3}{2} \exp[\cos x - 1].$$

Example 3.

Find the solution of the initial value problem

$$\frac{dy}{dx} + \exp(x^2)y = 0, \quad y(1) = 2.$$

Here $P(x) = \exp(x^2)$ so that

$$y(x) = 2 \exp\left(-\int_1^x \exp(s^2) ds\right).$$

Example 4.

Find the general solution of the linear differential equation

$$\frac{dy}{dx} - 2xy = x.$$

Here $P(x) = -2x$ so the integrating factor is given by

$$\mu(x) = \exp\left(\int P(x) dx\right) = \exp\left(-\int 2x dx\right) = \exp(-x^2).$$

Multiplying both sides of the equation by $\mu(x)$ we obtain the equivalent equation

$$\exp(-x^2) \left(\frac{dy}{dx} - 2xy\right) = x \exp(-x^2) \quad \text{or} \quad \frac{d}{dx} [\exp(-x^2)y] = x \exp(-x^2).$$

Hence,

$$\exp(-x^2)y = \int x \exp(-x^2) dx + c = -\frac{\exp(-x^2)}{2} + c.$$

The general solution is

$$y(x) = -\frac{1}{2} + c \exp(x^2).$$

Example 5.

Find the solution of the initial value problem

$$\frac{dy}{dx} + 2xy = x. \quad y(1) = 2.$$

Here $P(x) = 2x$ so that

$$\mu(x) = \exp\left(\int P(x) dx\right) = \exp\left(\int 2x dx\right) = \exp(x^2).$$

Multiplying both sides of the equation by $\mu(x)$ we obtain the equivalent equation

$$\exp(x^2) \left(\frac{dy}{dx} + 2xy\right) = x \exp(x^2) \quad \text{or} \quad \frac{d}{dx} [\exp(x^2)y] = x \exp(x^2).$$

Hence

$$\int_1^x \frac{d}{dt} [y(t) \exp(t^2)] dt = \int_1^x t \exp(t^2) dt$$

so that

$$y(t) \exp(t^2) \Big|_1^x = \frac{1}{2} \exp(t^2) \Big|_1^x.$$

Therefore,

$$y(x) \exp(x^2) - 2e = \frac{1}{2} \exp(x^2) - \frac{e}{2}$$

and

$$y(x) = \frac{1}{2} + \frac{3e}{2} \exp(-x^2) = \frac{1}{2} + \frac{3}{2} \exp(1 - x^2).$$

Example 6.

Find the solution of the initial value problem

$$\frac{dy}{dx} + y = \frac{1}{1+x^2}. \quad y(2) = 3.$$

Here $P(x) = 1$ so that

$$\mu(x) = \exp\left(\int P(x) dx\right) = \exp\left(\int 1 dx\right) = \exp(x).$$

Multiplying both sides of the equation by $\mu(x)$ we obtain the equivalent equation

$$\exp(x) \left(\frac{dy}{dx} + y\right) = \frac{\exp(x)}{1+x^2} \quad \text{or} \quad \frac{d}{dx} [\exp(x)y] = \frac{\exp(x)}{1+x^2}.$$

Hence

$$\int_2^x \frac{d}{dt} [y(t) \exp(t)] dt = \int_2^x \frac{\exp(t)}{1+t^2} dt$$

so that

$$y(x) \exp(x) - 3e^2 = \int_2^x \frac{\exp(t)}{1+t^2} dt$$

and

$$y(x) = \exp(-x) \left[3e^2 + \int_2^x \frac{\exp(t)}{1+t^2} dt \right].$$

Example 7.

Find the general solution of the linear differential equation

$$(1+x^2)dy + (xy + x^3 + x)dx = 0 \quad \text{or} \quad \frac{dy}{dx} + \left(\frac{x}{1+x^2}\right)y = -x, \quad (1)$$

An integrating factor is $\exp\left(\int \frac{x}{1+x^2} dx\right) = \sqrt{1+x^2}$ so that

$$\frac{d}{dx} \left[y\sqrt{1+x^2} \right] = -x\sqrt{1+x^2} \implies y = -\frac{1}{3}(1+x^2) + c(1+x^2)^{-\frac{1}{2}}. \quad (2)$$

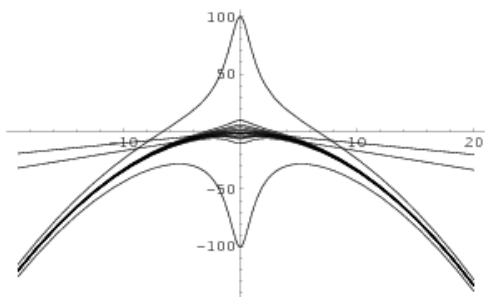


FIGURE 1. Graphs of $y = -\frac{1}{3}(1+x^2) + c(1+x^2)^{-\frac{1}{2}}$ for various choices of c .

Example 8.

The initial-value problem

$$\frac{dy}{dt} = \frac{y}{2y^2+t}, \quad y(1) = 5, \quad \text{or} \quad \frac{dt}{dy} - \frac{t}{y} = 2y, \quad t(5) = 1, \quad (3)$$

is linear in t (as a function of y). Its integrating factor is $1/y$, thus

$$\frac{d}{dy} \left[\frac{t}{y} \right] = 2 \implies t = 2y^2 + cy.$$

If $y(1) = 5$ then $c = -49/5$ and $t = 2y^2 - (49/5)y$.

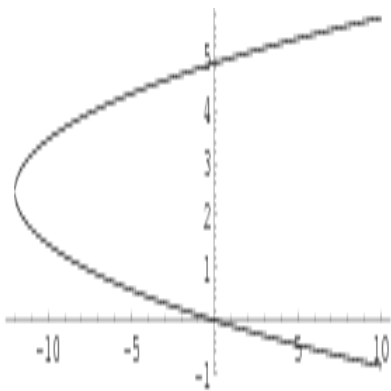


FIGURE 2. Graph of the integral curve $t = 2y^2 - (49/5)y$

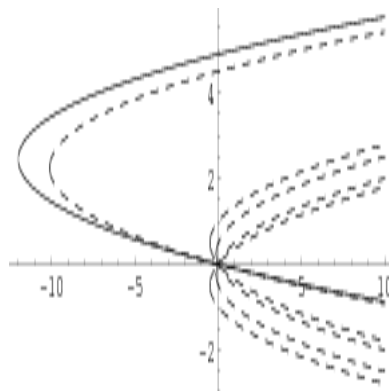


FIGURE 3. Other integral curves of $t = 2y^2 + cy$

The solutions of the initial value problem (3) and some other integral (dashed) curves of the differential equation

$$\frac{dy}{dt} = \frac{y}{2y^2+t}, \quad \text{are shown in Figure 3.}$$

Remark 1. Please observe that the graph in Figure 2 does not represent a function !!! Which branch of the graph represents the solution satisfying the initial condition $y(1) = 5$.