

FALLING OBJECT PROBLEM III – A MEDIUM WITH RESISTANCE
MATH 481A - SPRING 2026

An object of mass m is projected vertically downward with initial velocity equal to v_0 in a medium offering resistance proportional to the square root of the magnitude of the velocity.

- (a) Find a relation between the velocity v and the time t if the drag force equals $c\sqrt{v}$. Find the terminal with $c > 0$.
 (b) Find the terminal velocity of the object.

- (a) After choosing y -axis to be positive downward, Newton's law can be written as

$$m \frac{dv}{dt} = mg - c\sqrt{v}, \quad v(0) = v_0, \quad (1)$$

where $v(t)$ denotes the velocity at time t . Equation (1) is a separable differential equation; integration yields

$$\int \frac{1}{g - \frac{c}{m}\sqrt{v}} dv = \int dt \implies \left\{ g - \frac{c}{m}\sqrt{v} = w \right\} \implies -\frac{2m^2}{c^2} \int \frac{g-w}{w} dw = \int dt \implies -\frac{2m^2g}{c^2} \log|w| + \frac{2m^2}{c^2} w = t + C_0. \quad (2)$$

Using the initial condition, $v(0) = v_0$, the constant of integration C_0 in (2) is equal to

$$C_0 = -\frac{2m^2g}{c^2} \log \left| g - \frac{c}{m}\sqrt{v_0} \right| + \frac{2m^2}{c^2} \left(g - \frac{c}{m}\sqrt{v_0} \right), \quad (3)$$

and we have the following implicit relation between the velocity v and time t :

$$\frac{2m^2g}{c^2} \log \left| \frac{g - \frac{c}{m}\sqrt{v}}{g - \frac{c}{m}\sqrt{v_0}} \right| - \frac{2m}{c} (\sqrt{v} - \sqrt{v_0}) = t, \quad \text{or equivalently} \quad \frac{mg}{c} \log \left| \frac{mg - c\sqrt{v}}{mg - c\sqrt{v_0}} \right| + \sqrt{v} - \sqrt{v_0} = -\frac{c}{2m}t. \quad (4)$$

Equation (4) cannot be solved explicitly for v .

Remark 1

Expression (3), as well as equation (4), do **not** make sense when the initial value v_0 is equal to $\frac{m^2g^2}{c^2}$. In this case the solution to (1) is a constant function; i.e., the solution of initial value problem

$$m \frac{dv}{dt} = mg - c\sqrt{v}, \quad v(0) = \frac{m^2g^2}{c^2}, \quad (5)$$

is

$$v(t) = \frac{m^2g^2}{c^2}, \quad \text{for all } t \geq 0. \quad (6)$$

The solution given in (6) is certainly not physical one, indeed, it does not represent a motion of vertically downward projected object.

Remark 2

When the initial value in (1), $v_0 > \frac{m^2g^2}{c^2}$, the solution given by the implicit equation (4) is such that $v(t) > \frac{m^2g^2}{c^2}$. In fact, one can show that $v(t) \downarrow$ as a function of t . As above, this solution can not represent a motion of vertically downward projected object.

- (b) The second equation in (4) can be rewritten as

$$\left| \frac{mg - c\sqrt{v}}{mg - c\sqrt{v_0}} \right| = \exp \left(-\frac{c^2}{2m^2g}t \right) \exp \left(\frac{c}{mg}\sqrt{v_0} \right) \exp \left(-\frac{c}{mg}\sqrt{v} \right) \quad (7)$$

It easy to see that the right hand side of (7) converges to zero when $t \rightarrow \infty$ (*Do you know why ?*); thus, the left hand side must also converge to zero as $t \rightarrow \infty$. Since $v_0 \neq \frac{m^2g^2}{c^2}$, we must have

$$\lim_{t \rightarrow \infty} v(t) = \frac{m^2g^2}{c^2}. \quad (\text{the terminal velocity})$$

Remark 3

Using the fact that the solution of (1) (given in (4)) is unique, one can show that if $v_0 < \frac{m^2g^2}{c^2}$ then $v(t) \uparrow \frac{m^2g^2}{c^2}$ as $t \rightarrow \infty$. (*Can you provide the details for this argument ?*) Therefore, since $v(t) < \frac{m^2g^2}{c^2}$ for all t , we can rewrite the second equation in (4) as

$$\frac{mg}{c} \log \left(\frac{mg - c\sqrt{v}}{mg - c\sqrt{v_0}} \right) + \sqrt{v} - \sqrt{v_0} = -\frac{c}{2m}t.$$