FALLING OBJECT PROBLEM II – A MEDIUM WITH RESISTANCE
MATH 481A - SPRING 2026

A body of mass m falls from rest in a medium offering resistance proportional to the square of the velocity. Find velocity v(t) and compute the terminal velocity.

After choosing y-axis to be positive downward, Newton's law can be written as

$$m\frac{dv}{dt} = mg - cv^2, \qquad v(0) = 0, \tag{1}$$

where $v(t)$ denotes the velocity at time t. Equation (1) is a separable differential equation; integration yields

$$\frac{m}{c}\int_0^v \frac{dw}{\frac{mg}{c} - w^2} = \int_0^t ds \implies \frac{1}{\sqrt{\frac{mg}{c}}}\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) = \frac{c}{m}t \implies v(t) = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}t\right), \tag{2}$$

where

$$\tanh(x) = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}. \qquad \text{(see Figure 1, below, for the graph of tanh(x))} \tag{3}$$

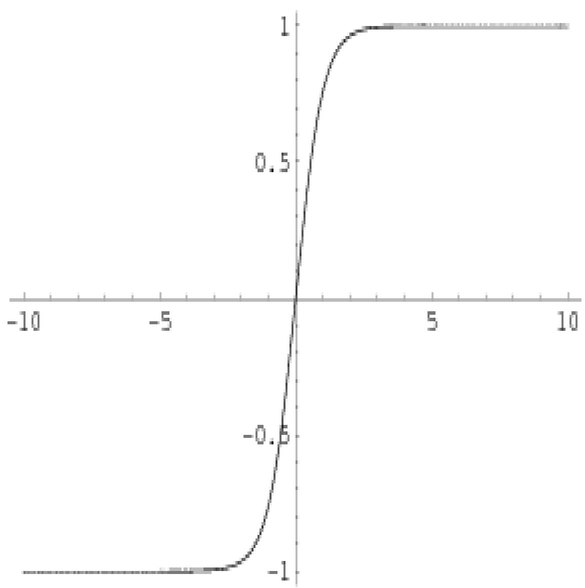

FIGURE 1. The graph of tanh(x).

Since $\lim_{x\to\infty} \tanh(x) = 1$, the limit of $v(t)$ as $t \to \infty$ is equal to

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{c}}.$$

Furthermore, one can show that $v(t) \uparrow \sqrt{\frac{mg}{c}}$ as $t \to \infty$.