Existence and uniqueness theorem

Math. 481a, Spring 2026

Theorem 1. *For $a < b$ and $c < d$, let $D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : a \leq x \leq b, c \leq d\}$ be a rectangular region in xy-plane. Assume that (x_0, y_0) is in the interior of D, i.e., $a < x_0 < b$ and $c < y_0 < d$.*

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on D, then there exists some interval $I_0 = (x_0 - h, x_0 + h) \subset [a, b]$, $h > 0$ on which there exists a unique solution $y(x)$ of

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

defined on I_0

Example 1

The largest region in xy-plane for which the differential equation

$$\frac{dy}{dx} = \sqrt{xy}, \qquad y(x_0) = y_0.$$

has a unique solution is

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy \geq 0,\ y \neq 0\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0,\ y > 0\} \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq 0,\ y < 0\}.$$

Indeed, D is the largest region of xy-plane where both $f(x, y) = \sqrt{xy}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{xy}}$ are continuous.

Remark: *Observe that $\sqrt{xy} \neq \sqrt{x}\sqrt{y}$ and thus, $\dfrac{x}{2\sqrt{xy}} \neq \dfrac{\sqrt{x}}{2\sqrt{y}}$.*

Example 2

Check that $y(x) = \tan(x + c)$ is a one-parameter family of solutions of the differential equation

$$y' = 1 + y^2.$$

Now, since $f(x, y) = 1 + y^2$ and $\dfrac{\partial f}{\partial y} = 2y$ are continuous everywhere in xy-plane. Theorem 1 guarantees that D can be taken to be the entire xy-plane.

At the same time, the initial value problem

$$y' = 1 + y^2, \qquad y(0) = 0,$$

is **NOT** defined on the interval $(-2, 2)$. Indeed, solving $y(0) = \tan c = 0$, we obtain $c = 0$ and the solution is $y(x) = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is **NOT** defined on $(-2, 2)$ because it contains . $\pm\,\pi/2$,

The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

Example 3

Check that $y(x) = -1/(x + c)$ is a one-parameter family of solutions of the differential equation

$$y' = y^2.$$

Now, since $f(x, y) = y^2$ and $\dfrac{\partial f}{\partial y} = 2y$ are continuous everywhere in xy-plane. Theorem 1 guarantees that D can be taken to be the entire xy-plane.

Now, $y(x) = 1/(2 - x)$ is a solution of the initial value

$$y' = y^2, \qquad y(1) = 1. \tag{1}$$

Indeed, applying $y(1) = 1$ to $y(x) = -1/(x + c)$ gives

$$1 = -\frac{1}{1 + c} \qquad \text{or} \qquad 1 + c = -1,$$

and $c = -2$. Thus, the solution is $y(x) = 1/(2 - x)$.

Similarly, one can check (*Do it!*) that $y(x) = 1/(2 - x)$ is a solution of the initial value

$$y' = y^2, \qquad y(3) = -1, \tag{2}$$

and yet these solutions are **NOT** the same.

Indeed, $y(x) = 1/(2 - x)$, a solution of (1), is defined on $(-\infty, 2)$, while $y(x) = 1/(2 - x)$, a solution of (2), is defined on $(2, \infty)$. *Do the graphs of these two solutions!!!*