Escape velocity problem

Math 481a, Spring 2026

A body of mass m is projected away from the earth in a direction perpendicular to the earth's surface with initial velocity v_0. Assuming that there is no air resistence, but taking into account the variation of the earth's gravitational field with distance, find the smallest initial velocity for which the body will not returm to earth. Such velocity is called the escape velocity.

Solution

We need first the expression for the weight of a body of mass m. At sea level it is equal to mg, however, we need to take into account variability of the earth's gravitational field with distance. From Newton's inverse-square law of gravitational attraction, we have

$$w(x) = \frac{K}{(R+x)^2},$$

where R is the radius of the earth and x is the altitude above sea level.

The constant K can be computed from the fact that at $x = 0$ (sea level), the weight is $w = mg$. Hence, $K = mgR^2$ and the expression for weight is

$$w(x) = \frac{mgR^2}{(R+x)^2}.$$

Please note that expanding $(R+x)^{-2}$ in a Taylor series about $x = 0$ we obtain

$$w(x) = mg\left(1 - 2\frac{x}{R} + \cdots\right).$$

Returning to the original problem, let the positive x-axis point away from the center of the earth along the line of a motion.

The differential equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0.$$

Note that we neglect drag forces in this problem, thus, except for the gravitational force, there are no other forces acting on the body. The minus sign indicates that $w(x)$ is directed in the negative x direction.

The right hand side of the above differential equation depends on x; thus it is convenient to think of x, rather than t, as the independent variable. In order to acomplish this change, we need to express $\frac{dv}{dt}$ in terms of $\frac{dv}{dx}$. This is done by using the chain rule.

We have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

Therefore, the differerential equation is replaced by

$$mv\frac{dv}{dx} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0.$$

Note that in the above equation v is now a function of x (and NOT t).

Separating variables and integrating we get

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

At $x = 0$, $v = v_0$ and $c = (v_0^2/2) - gR$, yielding

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

The plus sign for the body rising and the minus sign for the body falling.

Next, we can determine the maximum altitude ξ the body reaches when we set $v = 0$ and $x = \xi$ in the last expression for v.

We get

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}.$$

Solving the above equation for v_0 we find the initial velocity needed to lift the body to the altitude ξ:

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}}.$$

The escape velocity, v_e, is obtained by letting $\xi \to \infty$:

$$v_e = \sqrt{2gR}.$$