Problem 24 of Section 4.1

Math. 481a, Spring 2026

Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$

Solution

We have Taylor expansions:

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(x_0) - \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5);$$
$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{1}{2}h^2 f''(x_0) + \frac{1}{6}h^3 f'''(x_0) + \frac{1}{24}h^4 f^{(4)}(x_0) + O(h^5);$$
$$f(x_0+2h) = f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \frac{4}{3}h^3 f'''(x_0) + \frac{2}{3}h^4 f^{(4)}(x_0) + O(h^5);$$
$$f(x_0+3h) = f(x_0) + 3hf'(x_0) + \frac{9}{2}h^2 f''(x_0) + \frac{9}{2}h^3 f'''(x_0) + \frac{27}{8}h^4 f^{(4)}(x_0) + O(h^5);$$

Since the five-point formula uses $f(x_0-h)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$, expand $Af(x_0-h) + Bf(x_0+h) + Cf(x_0+2h) + Df(x_0+3h)$ in Taylor series using the above expansions:

$$\begin{aligned}
&Af(x_0-h) + Bf(x_0+h) + Cf(x_0+2h) + Df(x_0+3h) \\
&= f(x_0)(A+B+C+D) + f'(x_0)h\left[-A+B+2C+3D\right] + f''(x_0)h^2\left(\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D\right) \\
&+ f'''(x_0)h^3\left(-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D\right) + f^{(4)}(x_0)h^4\left(\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D\right) + O(h^5)
\end{aligned}$$

We want to eliminate the terms containing $f''(x_0)$, $f'''(x_0)$, and $f^{(4)}(x_0)$ and have the coefficient of $f'(x_0)$ equal 1. In order to accomplish that we need to solve the following system of linear equations:

$$-A + B + 2C + 3D = 1;$$
$$\frac{1}{2}A + \frac{1}{2}B + 2C + \frac{9}{2}D = 0;$$
$$-\frac{1}{6}A + \frac{1}{6}B + \frac{4}{3}C + \frac{9}{2}D = 0;$$
$$\frac{1}{24}A + \frac{1}{24}B + \frac{2}{3}C + \frac{27}{8}D = 0.$$

The solution is $A = -\frac{1}{4}$, $B = \frac{3}{2}$, $C = -\frac{1}{2}$, and $D = \frac{1}{12}$. Thus

$$-\frac{1}{4}f(x_0-h) + \frac{3}{2}f(x_0+h) - \frac{1}{2}f(x_0+2h) + \frac{1}{12}f(x_0+3h) = f(x_0)\left(-\frac{1}{4} + \frac{3}{2} - \frac{1}{2} + \frac{1}{12}\right) + hf'(x_0) + O(h^5)$$

Solving for $f'(x_0)$ gives

$$f'(x_0) = -\frac{1}{h}\left[\frac{10}{12}f(x_0) + \frac{1}{4}f(x_0-h) - \frac{3}{2}f(x_0+h) + \frac{1}{2}f(x_0+2h) - \frac{1}{12}f(x_0+3h)\right] + O(h^5),$$

which finally can be written as

$$f'(x_0) = \frac{1}{12h}\left[-f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)\right] + O(h^5).$$

Additional Practice Problem

Show that

$$\lim_{h\to 0} \frac{1}{12h}\left[-f(x_0-h) - 10f(x_0) + 18f(x_0+h) - 6f(x_0+2h) + f(x_0+3h)\right] = f'(x_0).$$