Math. 481a, Spring 2026

Additional examples on Richardson's extrapolation

Example 1

It can be shown that for $f \in C^\infty[a,b]$ the *composite Midpoint rule* can be written with an error term in the form

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots, \tag{1}$$

where $h = (b-a)/(n+2)$ and $x_j = a + (j+1)h$, $j = -1, 0, 1, \ldots, n+1$ and $n = 0, 1, 2, \ldots$.

The *Romberg integration* for *Midpoint Rule* uses the *composite Midpoint rule* (1) and *Richardson's extrapolation* to approximate $\int_a^b f(x)\,dx$. The results of the *composite Midpoint rule* with $n=0$: $R^M_{1,1}$ and then *composite Midpoint rule* with $n=2$: $R^M_{2,1}$ are used to compute $O(h^4)$ approximation:

$$R^M_{2,2} = R^M_{2.1} + \frac{1}{3}\left(R^M_{2,1} - R^M_{1,1}\right). \qquad \text{(Richardson's extrapolation)}$$

$$R^M_{1.1} = 2hf(x_0) \qquad \text{where } x_{-1} = a,\ x_0 = a+h,\ x_1 = b, \text{ and } h = (b-a)/2.$$

$$R^M_{2.1} = 2(h/2)\left[f(x_0) + f(x_2)\right],$$

where $x_{-1} = a$, $x_0 = a + h/2$, $x_1 = x_0 + h/2 = a + h$, $x_2 = x_1 + h/2 = a + 3h/2$, $x_3 = x_2 + h/2 = a + 2h = b$, and $h = (b-a)/2$.

We have

$$\begin{aligned} R^M_{2,2} &= R^M_{2.1} + \frac{1}{3}\left(R^M_{2,1} - R^M_{1,1}\right) = \frac{4}{3}R^M_{2,1} - \frac{1}{3}R^M_{1,1} \\ &= \frac{4}{3}h\left[f(a+h/2) + f(a+3h/2)\right] - 2(h/3)f(a+h). \end{aligned}$$

We let $\bar{h} = h/2$, and $R^M_{2,2}$ takes the form of the *Midpoint rule* with $n = 2$ (see formula (4.31) on page 199 of the textbook, 10th edition):

$$R^M_{2,2} = \frac{4}{3}\bar{h}\left[2f(a+\bar{h}) + 2f(a+3\bar{h}) - f(a+2\bar{h})\right] = \frac{4}{3}\bar{h}\left[2f(x_0) - f(x_1) + 2f(x_2)\right]$$

with $\bar{h} = (b-a)/4$.

Example 2

It can be shown that for $f \in C^\infty[a,b]$ *Second Derivative Midpoint Formula* can be written with an error term in the form

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right] + K_1h^2 + K_2h^4 + K_3h^6 + \ldots.$$

Use Richardson's extrapolation to obtain $O(h^4)$ formula.

Starting with $N_1(h) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right]$, we obtain

$$\begin{aligned} N_2(h) &= \frac{4}{3}N_1(h/2) - \frac{1}{3}N_1(h) \\ &= \frac{16}{3}\frac{1}{h^2}\left[f(x_0-h/2) - 2f(x_0) + f(x_0+h/2)\right] - \frac{1}{3}\frac{1}{h^2}\left[f(x_0+h) - 2f(x_0) + f(x_0+h)\right]. \end{aligned}$$

Using $\bar{h} = h/2$, after simplifications, we obtain

$$f''(x_0) \approx \frac{1}{12}\frac{1}{\bar{h}^2}\left[16f(x_0-\bar{h}) - 30f(x_0) + 16f(x_0+\bar{h}) - f(x_0-2\bar{h}) - f(x_0+2\bar{h})\right]. \tag{2}$$

For example, for $f(x) = \exp(x^2)$, and $x_0 = 0$, $f''(x_0) = 2$, and with $h = 0.1$, $N_1(0.1) = 2.0100334$. The error is 0.0100334, which is of order h^2. On the other hand, the approximation in (2) gives $f''(0) \approx 1.999864933$, with the error, 0.000135067, of order h^4.

Example 3

From calculus we know that

$$\lim_{h\to 0}(1+h)^{1/h} = e.$$

Additionally it can be shown that

$$(1+h)^{1/h} = e + \frac{1}{2}eh + \frac{11}{24}eh^2 - \frac{7}{16}eh^3 + \frac{2447}{5760}eh^4 - \frac{959}{2304}eh^5 + \ldots.$$

Use Richardson's extrapolation to obtain $O(h^2)$ approximation of e for $h = 0.1$.

$N_1(h) = (1+h)^{1/h}$ and for $h = 0.1$ we get 2.593742460 with the error of 0.124539368, while with

$$N_2(h) = \frac{4}{3}N_1(h/2) - \frac{1}{3}N_1(h) = \frac{4}{3}(1+h/2)^{2/h} - \frac{1}{3}(1+h)^{1/h},$$

$N_2(0.1) = 2.673149453$, with the error 0.045132375.