Math. 481a, Spring 2026

Divided differences

If $P_n(x)$ is Lagrange interpolating polynomial that agrees with f at the distinct $x_0, x_1, \ldots, x_n$, the divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants to be computed.

We have

$$a_0 = P_n(x_0) = f(x_0) \overset{def}{=} f[x_0]. \tag{2}$$

Since

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1),$$

we obtain

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \overset{def}{=} f[x_0, x_1]. \tag{3}$$

The **zeroth divided difference** of f with respect to x_i is the value of f at x_i:

$$f[x_i] = f(x_i).$$

In particular, a_0, in (2), is equal to $f[x_0]$.

The **first divided difference** of f with respect to x_i and x_{i+1} is

$$f[x_i, x_{i+1}] \overset{def}{=} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

In particular, a_1, in (3), is equal to $f[x_0, x_1]$.

The **second divided difference** of f with respect to x_i, x_{i+1}, x_{i+2} is

$$f[x_i, x_{i+1}, x_{i+2}] \overset{def}{=} \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}. \qquad \left(= \frac{\frac{f(x_{i+2}) - f(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}{x_{i+2} - x_i} \right)$$

And inductively, if the divided differences $f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}]$ have been determined, the **kth divided difference** with respect to $x_i, x_{i+1}, \ldots, x_{i+k}$ is given by

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] \overset{def}{=} \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}] - f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

We have

$$f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1),$$

and thus, since $a_0 = f[x_0] = f(x_0)$, and $a_1 = f[x_0, x_1]$,

$$\begin{aligned}
a_2 &= \frac{f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{f(x_2) - f(x_1) + f(x_1) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{f[x_1, x_2]}{x_2 - x_0} + \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)} \\
&= \frac{f[x_1, x_2]}{x_2 - x_0} - \frac{f[x_0, x_1]}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\
&= f[x_0, x_1, x_2].
\end{aligned}$$

In general, a_k in (1) are given by

$$a_k = f[x_0, x_1, \ldots, x_k], \qquad k = 0, 1, \ldots, n,$$

and $P_n(x)$ in (1) can be rewritten as

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1}).$$

Example

$$\begin{aligned} x_0 &= 0 & f(x_0) &= 0 \\ x_1 &= 1 & f(x_1) &= 2 \\ x_2 &= 2 & f(x_2) &= 5 \end{aligned}$$

The Lagrange interpolating polynomial that agrees with f at x_0, x_1, x_2 is

$$P_2(x) = \frac{1}{2}x^2 + \frac{3}{2}x.$$

Now,

$$a_0 = f[x_0] = P_n(x_0) = f(0) = 0,$$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2-0}{1-0} = 2,$$

Since,

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{5-2}{2-1} = 3,$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3-2}{2-0} = \frac{1}{2},$$

and

$$P_2(x) = \frac{1}{2}x^2 + \frac{3}{2}x = a_0 + a_1 x + a_2 x(x-1) = 0 + 2x + \frac{1}{2}x(x-1).$$