7.2 The Natural Logarithmic and Exponential Functions

Your understanding of logarithms and exponentials as algebraic operations is important, and it will be put to use in the coming pages. For example, you may recall the following relationship that will be used frequently: If b denotes a *base* with $b > 0$ and $b \neq 1$, then

$$y = b^x \quad \text{if and only if} \quad x = \log_b y.$$

However, to do calculus with logarithms and exponentials, we must view them not just as operations, but as functions. Once we define logarithmic and exponential *functions*, many important questions quickly follow.

- What are the domains of b^x and $\log_b x$?
- How do we assign a meaning to expressions such as 2^π or $\log_3 \pi$?
- Are these functions continuous on their domains?
- What are their derivatives?
- What new integrals can be evaluated using these functions?

It all begins with the *natural logarithmic function*, which is defined in terms of a definite integral, after which we use the theory of inverse functions (Section 7.1) to develop the *natural exponential function*. Our objective in this section is to place these important functions on a solid foundation by presenting a rigorous development of their properties.

Before embarking on this program, we offer a roadmap to help guide you through the section. We carry out the following three steps.

1. We first define the natural logarithmic function, denoted $\ln x$, in terms of an integral, and then derive the properties of $\ln x$ directly from this new definition.
2. Next, the natural exponential function e^x is introduced as the inverse of $\ln x$, and the properties of e^x are developed by appealing to this inverse relationship. We also present derivative and integral formulas associated with these functions.
3. Finally, we define the general exponential function b^x in terms of e^x so that two crucial properties of the natural logarithmic and exponential functions can be extended to all real numbers. One of these properties is used to derive a limit definition of e that is used to approximate e.

After establishing the properties of the natural logarithmic and exponential functions, we conclude the section with derivative and integral formulas associated with e^x, and we present the technique of logarithmic differentiation.

Note »

> Logarithms were invented around 1600 for calculating purposes by the Scotsman John Napier and the Englishman Henry Briggs. Unfortunately, the word *logarithm*, derived from the Greek for reasoning (*logos*) with numbers (*arithmos*), doesn't help with the meaning of the word. **When you see *logarithm*, you should think *exponent*.**

Step 1: The Natural Logarithm »

Our aim is to develop the properties of the natural logarithm using definite integrals. We begin with the following definition.

> **DEFINITION** **The Natural Logarithm**
>
> The **natural logarithm** of a number $x > 0$ is $\ln x = \displaystyle\int_1^x \frac{1}{t}\, dt.$

All the familiar geometric and algebraic properties of the natural logarithmic function follow directly from this integral definition.

Properties of the Natural Logarithm

Domain, range, and sign Because the natural logarithm is defined as a definite integral, its value is the net area under the curve $y = \frac{1}{t}$ between $t = 1$ and $t = x$. The integrand is undefined at $t = 0$, so the domain of $\ln x$ is $(0, \infty)$. On the interval $(1, \infty)$, $\ln x$ is positive because the net area of the region under the curve is positive (**Figure 7.14**). On $(0, 1)$, we have $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$, which implies $\ln x$ is negative. As expected, when $x = 1$, we have $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$. The net area interpretation of $\ln x$ also implies that the range of $\ln x$ is $(-\infty, \infty)$ (see Exercise 112 for an outline of a proof).

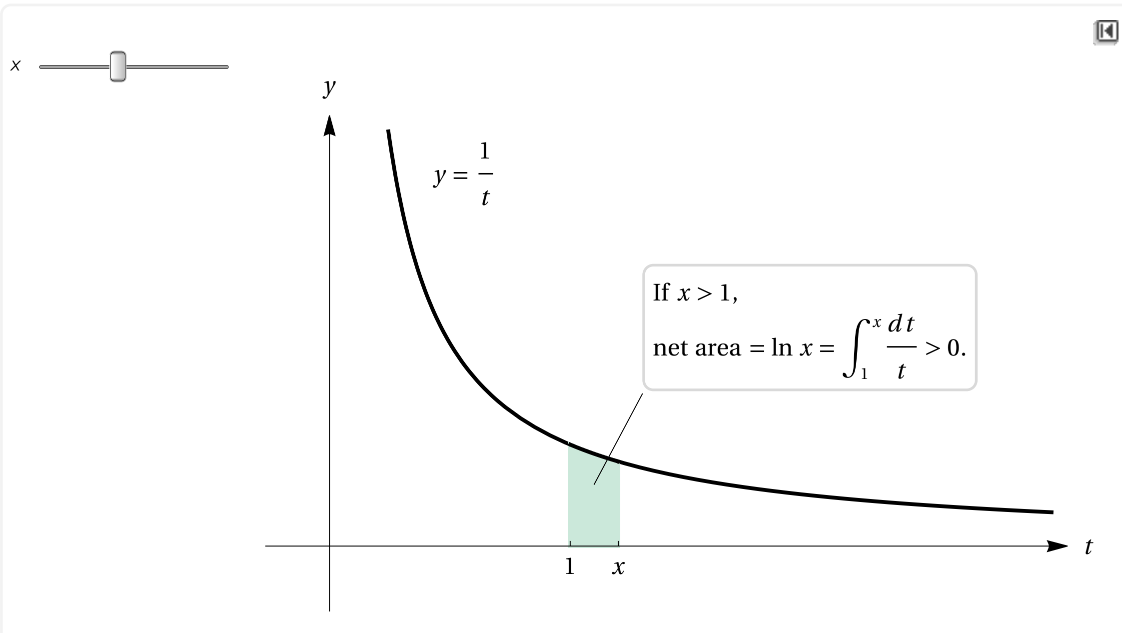


Figure 7.14

Derivative The derivative of the natural logarithm follows immediately from its definition and the Fundamental Theorem of Calculus:

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}, \quad \text{for } x > 0.$$

Note »

By the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We have two important consequences:

- Because its derivative is defined for $x > 0$, $\ln x$ is differentiable for $x > 0$, which means it is continuous on its domain (Theorem 3.1).

- Because $\frac{1}{x} > 0$ for $x > 0$, $\ln x$ is strictly increasing and one-to-one on its domain; therefore, it has a well-defined inverse.

The Chain Rule allows us to extend the derivative property to all nonzero real numbers (Exercise 110). By differentiating $\ln(-x)$ for $x < 0$, we find that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \text{ for } x \neq 0.$$

More generally, by the Chain Rule,

$$\frac{d}{dx}(\ln|u(x)|) = \frac{d}{du}(\ln|u(x)|)\, u'(x) = \frac{u'(x)}{u(x)}.$$

Quick Check 1 What is the domain of $\ln|x|$? ◆

Answer »

$\{x : x \neq 0\}$

Graph of ln x As noted before, $\ln x$ is continuous and strictly increasing for $x > 0$. The second derivative, $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2}$, is negative for $x > 0$, which implies the graph of $\ln x$ is concave down for $x > 0$. As demonstrated in Exercise 112,

$$\lim_{x\to\infty} \ln x = \infty, \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty.$$

This information, coupled with the fact that $\ln 1 = 0$, gives the graph of $y = \ln x$ **(Figure 7.15)**.

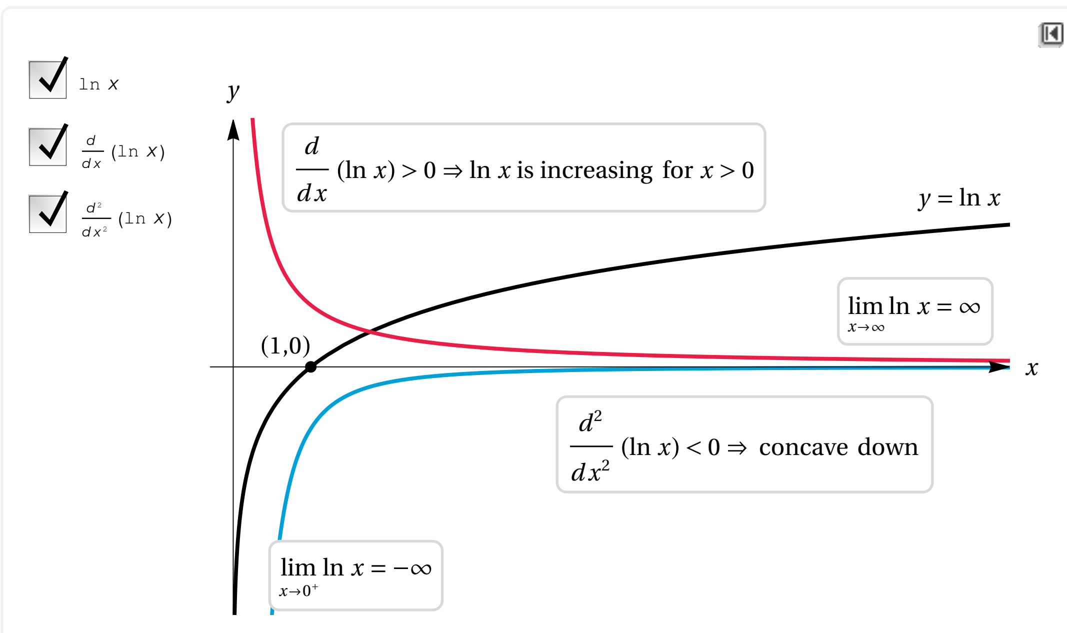


Figure 7.15

The graphs of $y = \ln x$, $y = \ln|x|$, and their derivatives are shown in **Figure 7.16**.

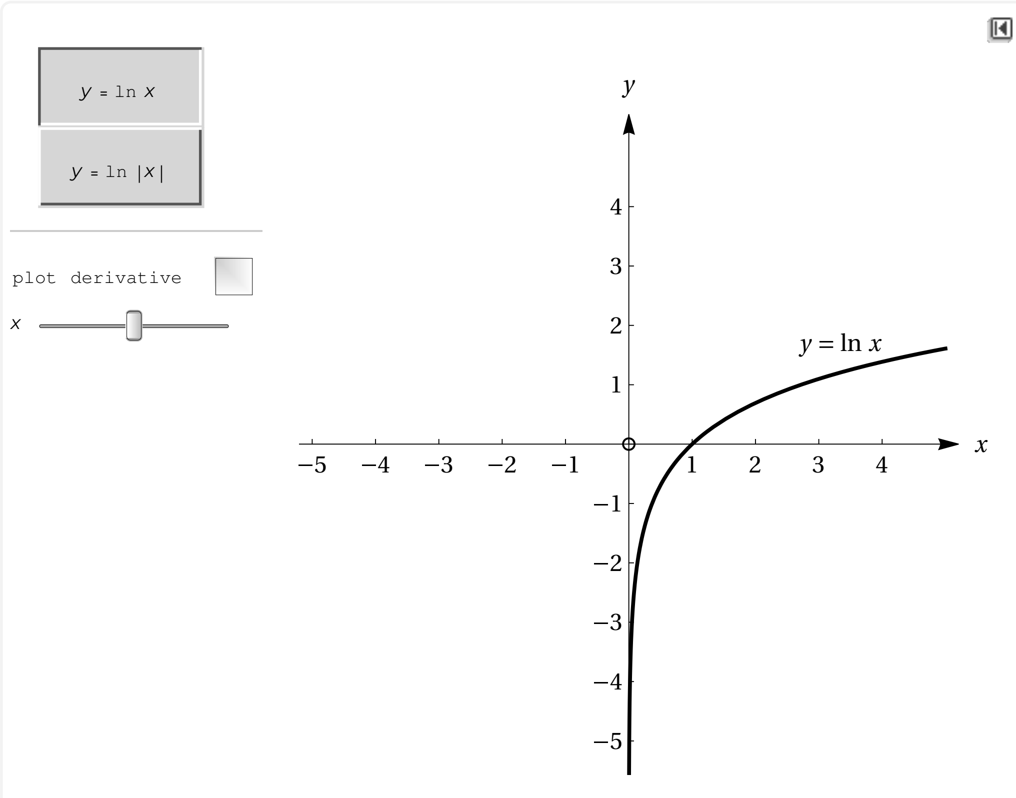


Figure 7.16

Logarithm of a product The familiar logarithm property

$$\ln xy = \ln x + \ln y, \text{ for } x > 0,\ y > 0$$

may be proved using the integral definition:

$$\begin{aligned}
\ln xy &= \int_1^{xy} \frac{dt}{t} && \text{Definition of } \ln xy \\
&= \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} && \text{Additive property of integrals} \\
&= \int_1^{x} \frac{dt}{t} + \int_1^{y} \frac{du}{u} && \text{Substitute } u = \frac{t}{x} \text{ in second integral.} \\
&= \ln x + \ln y. && \text{Definition of the natural logarithm}
\end{aligned}$$

Logarithm of a quotient Assuming $x > 0$ and $y > 0$, the product property and a bit of algebra give

$$\ln x = \ln\left(y \cdot \frac{x}{y}\right) = \ln y + \ln \frac{x}{y}.$$

Solving for $\ln \frac{x}{y}$, we have

$$\ln \frac{x}{y} = \ln x - \ln y,$$

which is the quotient property for logarithms. (Also see Exercise 72.)

Logarithm of a power Assuming $x > 0$ and p is an integer, we have

$$\begin{aligned} \ln x^p &= \int_1^{x^p} \frac{dt}{t} && \text{Definition of } \ln x^p \\ &= p\int_1^x \frac{du}{u} && \text{Let } t = u^p;\ dt = p\,u^{p-1}\,du. \\ &= p\ln x. && \text{By definition, } \ln x = \int_1^x \frac{du}{u}. \end{aligned}$$

This argument relies on the Power Rule ($dt = p\,u^{p-1}\,du$), which we proved only for integer exponents in Sections 3.3 and 3.4. Later in this section, we prove that $\ln x^p = p \ln x$ for all real values of p.

Integrals Because $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$, we have

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

We have shown that the familiar properties of $\ln x$ follow from its integral definition.

THEOREM 7.4 Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln(xy) = \ln x + \ln y$, for $x > 0$, $y > 0$
3. $\ln \dfrac{x}{y} = \ln x - \ln y$, for $x > 0$, $y > 0$
4. $\ln x^p = p\ln x$, for $x > 0$ and p an integer
5. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$, for $x \neq 0$
6. $\dfrac{d}{dx}(\ln|u(x)|) = \dfrac{u'(x)}{u(x)}$, for $u(x) \neq 0$
7. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

EXAMPLE 1 Derivatives involving $\ln x$

Find $\dfrac{dy}{dx}$ for the following functions.

a. $y = \ln 4x$

b. $y = x\ln x$

c. $y = \ln|\sec x|$

d. $y = \dfrac{\ln x^2}{x^2}$

SOLUTION »

a. Using the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(\ln 4x) = \frac{1}{4x} \cdot 4 = \frac{1}{x}.$$

An alternative method uses a property of logarithms before differentiating:

$$\begin{aligned} \frac{d}{dx}(\ln 4x) &= \frac{d}{dx}(\ln 4 + \ln x) && \ln xy = \ln x + \ln y \\ &= 0 + \frac{1}{x} = \frac{1}{x}. && \ln 4 \text{ is a constant.} \end{aligned}$$

Note »

Because $\ln x$ and $\ln 4x$ differ by a constant ($\ln 4x = \ln x + \ln 4$), the derivatives of $\ln x$ and $\ln 4x$ are equal.

b. By the Product Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

c. Using property 6 of Theorem 7.4,

$$\frac{dy}{dx} = \frac{1}{\sec x}\left(\frac{d}{dx}(\sec x)\right) = \frac{1}{\sec x}(\sec x \tan x) = \tan x.$$

d. The Quotient Rule and Chain Rule give

$$\frac{dy}{dx} = \frac{x^2\left(\frac{1}{x^2} \cdot 2x\right) - \left(\ln x^2\right) 2x}{\left(x^2\right)^2} = \frac{2x - 2x \ln x^2}{x^4} = \frac{2\left(1 - \ln x^2\right)}{x^3}.$$

Related Exercises 17–18, 21–22 ◆

Quick Check 2 Find $\dfrac{d}{dx}(\ln x^p)$, where $x > 0$ and p is an integer, in two ways: (1) using the Chain Rule and (2) by first using a property of logarithms. ◆

Answer »

$\dfrac{p}{x}$

EXAMPLE 2 Integrals with ln *x*

Evaluate $\displaystyle\int_0^4 \frac{x}{x^2 + 9}\, dx.$

SOLUTION »

$$\begin{aligned}
\int_0^4 \frac{x}{x^2+9}\,dx &= \frac{1}{2}\int_9^{25}\frac{du}{u} && \text{Let } u = x^2+9;\ du = 2x\,dx.\\
&= \frac{1}{2}\ln|u|\Big|_9^{25} && \text{Fundamental Theorem}\\
&= \frac{1}{2}(\ln 25 - \ln 9) && \text{Evaluate.}\\
&= \ln\frac{5}{3} && \text{Properties of logarithms}
\end{aligned}$$

Related Exercise 44 ◆

Step 2: The Exponential Function »

We have established that $f(x) = \ln x$ is a continuous, increasing function on the interval $(0, \infty)$. Therefore, it is one-to-one and its inverse function exists on $(0, \infty)$. We denote the inverse function $f^{-1}(x) = \exp(x)$. Its graph is obtained by reflecting the graph of $f(x) = \ln x$ about the line $y = x$ (**Figure 7.17**). The domain of $\exp(x)$ is $(-\infty, \infty)$ because the range of $\ln x$ is $(-\infty, \infty)$, and the range of $\exp(x)$ is $(0, \infty)$ because the domain of $\ln x$ is $(0, \infty)$.

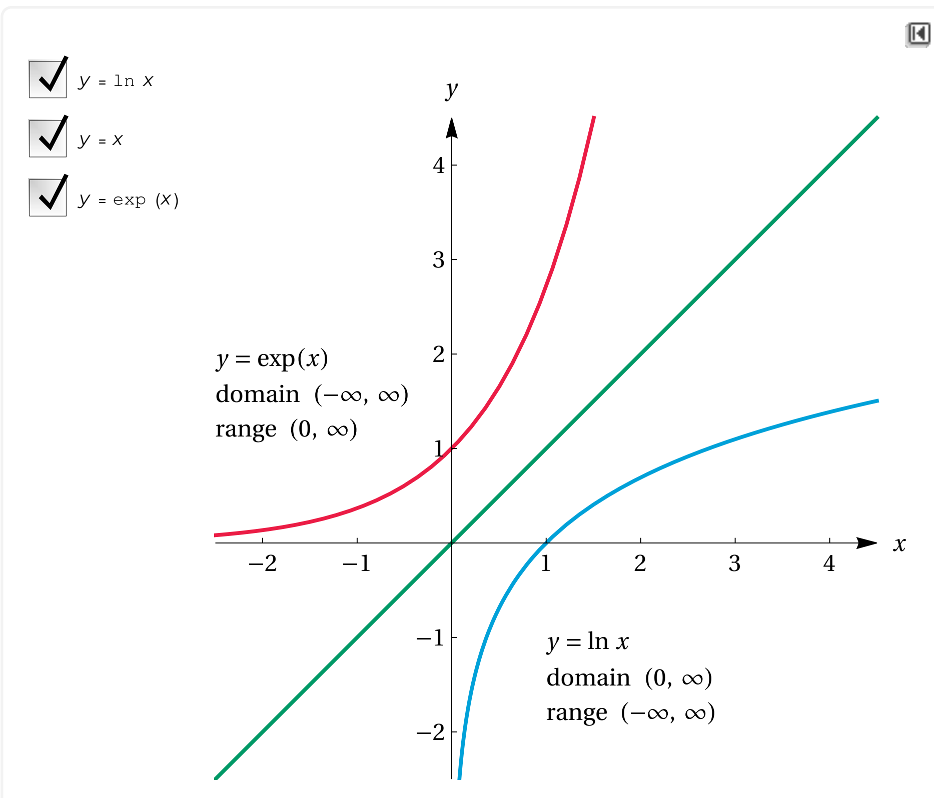


Figure 7.17

The usual relationships between a function and its inverse also hold:

- $y = \exp(x)$ if and only if $x = \ln y$
- $\exp(\ln x) = x$, for $x > 0$, and $\ln(\exp(x)) = x$, for all x

We now appeal to the properties of $\ln x$ and use the inverse relations between $\ln x$ and $\exp(x)$ to show that

$\exp(x)$ satisfies the properties of any exponential function. For example, if $x_1 = \ln y_1$ and $x_2 = \ln y_2$, then it follows that $y_1 = \exp(x_1)$, $y_2 = \exp(x_2)$, and

$$\begin{aligned} \exp(x_1 + x_2) &= \exp(\underbrace{\ln y_1 + \ln y_2}_{\ln y_1 y_2}) && \text{Substitute } x_1 = \ln y_1,\ x_2 = \ln y_2. \\ &= \exp(\ln y_1 y_2) && \text{Properties of logarithms} \\ &= y_1 y_2 && \text{Inverse property of } \exp(x) \text{ and } \ln x \\ &= \exp(x_1)\exp(x_2). && y_1 = \exp(x_1),\ y_2 = \exp(x_2) \end{aligned}$$

Therefore, $\exp(x)$ satisfies the property of exponential functions $b^{x_1 + x_2} = b^{x_1} b^{x_2}$. Similar arguments show that $\exp(x)$ satisfies other characteristic properties of all exponential functions (Exercise 111):

$$\begin{aligned} \exp(0) &= 1, \\ \exp(x_1 - x_2) &= \frac{\exp(x_1)}{\exp(x_2)}, \text{ and} \\ (\exp(x))^p &= \exp(p\,x), \text{ for integers } p. \end{aligned}$$

Suspecting that $\exp(x)$ is an exponential function, we proceed to identify its base. Let's consider the real number $\exp(1)$, and with a bit of forethought, call it e. The inverse relationship between $\ln x$ and $\exp(x)$ implies that

$$\text{if } e = \exp(1), \text{ then } \ln e = \ln(\exp(1)) = 1.$$

Using the fact that $\ln e = 1$ and the integral definition of $\ln x$, we now formally define e.

DEFINITION **The Number e**

The number e is the real number that satisfies $\ln e = \int_1^e \frac{dt}{t} = 1$.

Note »

The constant e was identified and named by the Swiss mathematician Leonhard Euler (1707-1783) (pronounced "oiler").

The number e has the property that the area of the region bounded by the graph of $y = \frac{1}{t}$ and the t-axis on the interval $[1, e]$ is 1 (**Figure 7.18**). Note that $\ln 2 < 1$ and $\ln 3 > 1$ (Exercise 113). Because $\ln x$ is continuous on its domain, the Intermediate Value Theorem ensures that there is a number e with $2 < e < 3$ such that $\ln e = 1$.

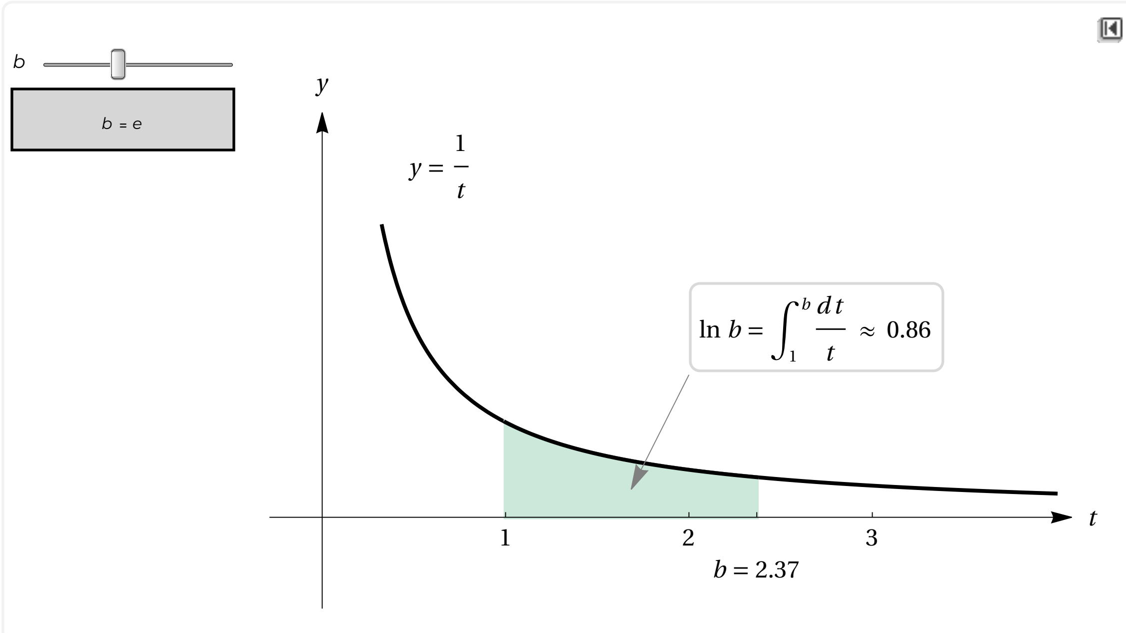


Figure 7.18

We can now show that indeed $\exp(x)$ is the exponential function e^x. Assume that p is an integer and note that $e^p > 0$. By property 4 of Theorem 7.4, we have

$$\ln e^p = p\underbrace{\ln e}_{1} = p.$$

Using the inverse relationship between $\ln x$ and $\exp(x)$, we also know that

$$\ln \exp(p) = p.$$

Equating these two expressions for p, we conclude that $\ln e^p = \ln \exp(p)$. Because $\ln x$ is a one-to-one function, it follows that

$$e^p = \exp(p), \text{ for integers } p,$$

and we conclude that $\exp(x)$ is the exponential function with base e. We already know how to evaluate e^x when x is rational. For example, $e^3 = e \cdot e \cdot e$, $e^{-2} = \frac{1}{e \cdot e}$, and $e^{1/2} = \sqrt{e}$. But how do we evaluate e^x when x is irrational? We proceed as follows. The function $x = \ln y$ is defined for $y > 0$, and its range is all real numbers. Therefore, the domain of its inverse $y = \exp(x)$ is all real numbers; that is $\exp(x)$ is defined for all real numbers. We now define e^x to be $\exp(x)$ when x is irrational.

DEFINITION **The Exponential Function**

For real numbers x, $y = e^x = \exp(x)$, where $x = \ln y$.

We may now dispense with the notation $\exp(x)$ and use e^x as the inverse of $\ln x$. The usual inverse relationships between e^x and $\ln x$ hold, and the properties of $\exp(x)$ can now be written for e^x.

THEOREM 7.5 Properties of e^x

The exponential function e^x satisfies the following properties, all of which result from the integral definition of $\ln x$. Let x and y be any real numbers.

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = \dfrac{e^x}{e^y}$
3. $(e^x)^p = e^{xp}$, where p is an integer
4. $\ln(e^x) = x$, for all x
5. $e^{\ln x} = x$, for $x > 0$

Step 3: General Exponential Functions »

It is now a short step to define the exponential function b^x for positive bases with $b \neq 1$ and for all real numbers x. By properties 3 and 5 of Theorem 7.5, if x is an integer, then

$$b^x = \underbrace{\left(e^{\ln b}\right)}_{b}{}^{x} = e^{x \ln b};$$

this important relationship expresses b^x in terms of e^x. Because e^x is defined for all real x, we use this relationship to define b^x for all real x.

DEFINITION Exponential Functions with General Bases

Let b be a positive real number with $b \neq 1$. Then for all real x,

$$b^x = e^{x \ln b}.$$

This definition comes with an immediate and important consequence. We use the definition of b^x to write

$$x^p = e^{p \ln x}, \text{ for } x > 0 \text{ and } p \text{ real}.$$

Taking the natural logarithm of both sides and using the inverse relationship between e^x and $\ln x$, we find that

$$\ln x^p = \ln e^{p \ln x} = p \ln x, \text{ for } x > 0 \text{ and } p \text{ real}.$$

In this way, we extend property 4 of Theorem 7.4 to real powers.

Note »

Knowing that $\ln x^p = p \ln x$, for real p, we can also extend property 3 of Theorem 7.5 to real numbers. For real x and y, we take the natural logarithm of both sides of $z = (e^x)^y$, which gives $\ln z = y \ln e^x = x y$, or $z = e^{xy}$. Therefore, $(e^x)^y = e^{xy}$.

Quick Check 3 Simplify $e^{\ln 2x}$, $\ln\left(e^{2x}\right)$, $e^{2 \ln x}$, $\ln(2e^x)$. ◆

Answer »

$2x$, $2x$, x^2, $x + \ln 2$

Approximating *e* »

We have shown that the number e serves as a base for both $\ln x$ and e^x, but how do we approximate its value? Recall that the derivative of $\ln x$ at $x = 1$ is 1. By the definition of the derivative, it follows that

$$\begin{aligned} 1 &= \frac{d}{dx}(\ln x)\Big|_{x-1} \\ &= \lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} && \text{Derivative of } \ln x \text{ at } x = 1 \\ &= \lim_{h\to 0}\frac{\ln(1+h)}{h} && \ln 1 = 0 \\ &= \lim_{h\to 0}\ln(1+h)^{1/h}. && p\ln x = \ln x^p \end{aligned}$$

Note »

> Because $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\frac{d}{dx}(\ln x)\Big|_{x=1} = \frac{1}{1} = 1$.

The natural logarithm is continuous for $x > 0$, so it is permissible to interchange the order of $\lim\limits_{h\to 0}$ and the evaluation of $\ln(1+h)^{1/h}$. The result is that

$$\ln\underbrace{\left(\lim_{h\to 0}(1+h)^{1/h}\right)}_{e} = 1.$$

Note »

> Here we rely on a strong version of Theorem 2.10 of Section 2.6. If f is continuous at $g(a)$ and $\lim\limits_{x\to a} g(x)$ exists, then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)$.

Observe that the limit within the bracket is e because $\ln e = 1$ and only one number satisfies this equation. Therefore, we have isolated e as a limit:

$$e = \lim_{h\to 0}(1+h)^{1/h}.$$

It is evident from the values in Table 7.2 that $(1+h)^{1/h} \to 2.718282\ \ldots$ as $h \to 0$. The value of this limit is e, and it has been computed to millions of digits. A better approximation,

$$e \approx 2.718281828459045\ ,$$

is obtained by methods introduced in Chapter 11.

Table 7.2

h	$(1+h)^{1/h}$	h	$(1+h)^{1/h}$
10^{-1}	2.593742	-10^{-1}	2.867972
10^{-2}	2.704814	-10^{-2}	2.731999
10^{-3}	2.716924	-10^{-3}	2.719642
10^{-4}	2.718146	-10^{-4}	2.718418
10^{-5}	2.718268	-10^{-5}	2.718295
10^{-6}	2.718280	-10^{-6}	2.718283
10^{-7}	2.718282	-10^{-7}	2.718282

Derivatives and Integrals »

The derivative of the exponential function follows directly from Theorem 7.3 (derivatives of inverse functions) or by using the Chain Rule. Taking the latter course, we observe that $\ln(e^x) = x$ and then differentiate both sides with respect to x:

$$\frac{d}{dx}(\ln e^x) = \underbrace{\frac{d}{dx}(x)}_{1}$$

$$\frac{1}{e^x}\frac{d}{dx}(e^x) = 1 \qquad \frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)} \text{ (Chain Rule)}$$

$$\frac{d}{dx}(e^x) = e^x. \qquad \text{Solve for } \frac{d}{dx}(e^x).$$

We obtain the remarkable result that the exponential function is its own derivative, which implies that the line tangent to the graph of $y = e^x$ at $(0, 1)$ has slope 1 (**Figure 7.19**).

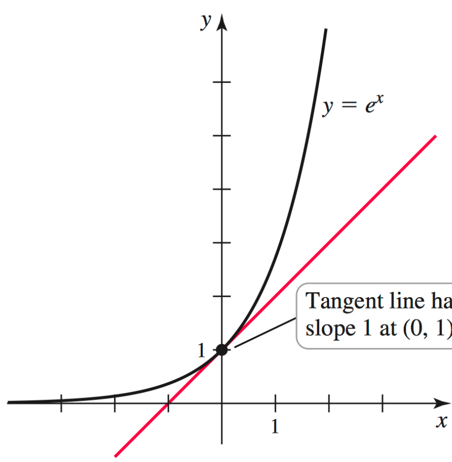


Figure 7.19

It immediately follows that e^x is its own antiderivative up to a constant; that is,

$$\int e^x\,dx = e^x + C.$$

Extending these results using the Chain Rule, we have the following theorem.

THEOREM 7.6 Derivative and Integral of the Exponential Function

For real numbers x,

$$\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)}\,u'(x) \quad \text{and} \quad \int e^x\,dx = e^x + C.$$

Note »

As shown in Example 5a, the integral formula in Theorem 7.6 can be generalized: $\int e^{a\,x}\,dx = \frac{1}{a}\,e^{a\,x} + C$.

Quick Check 4 What is the slope of the curve $y = e^x$ at $x = \ln 2$? What is the area of the region bounded by the graph of $y = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$? ◆

Answer »

EXAMPLE 3 Derivatives involving exponential functions

Evaluate the following derivatives.

a. $\frac{d}{dx}\left(3\,e^{2\,x} - 4\,e^x + e^{-3\,x}\right)$

b. $\frac{d}{dt}\left(\frac{e^t}{e^{2\,t} - 1}\right)$

c. $\left.\frac{d}{dx}\left(e^{\cos\pi x}\right)\right|_{x=1/2}$

SOLUTION »

a.

$$\begin{aligned}
\frac{d}{dx}\left(3\,e^{2\,x} - 4\,e^x + e^{-3\,x}\right) &= 3\,\frac{d}{dx}\left(e^{2\,x}\right) - 4\,\frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{-3\,x}\right) && \text{Sum and Constant Multiple Rules}\\
&= 3\cdot 2\cdot e^{2\,x} - 4\,e^x + (-3)\,e^{-3\,x} && \text{Chain Rule}\\
&= 6\,e^{2\,x} - 4\,e^x - 3\,e^{-3\,x} && \text{Simplify.}
\end{aligned}$$

b.

$$\frac{d}{dt}\left(\frac{e^t}{e^{2\,t} - 1}\right) = \frac{\left(e^{2\,t} - 1\right)e^t - e^t\cdot 2\,e^{2\,t}}{\left(e^{2\,t} - 1\right)^2} = -\frac{e^{3\,t} + e^t}{\left(e^{2\,t} - 1\right)^2} \quad \text{Quotient Rule}$$

c. First note that by the Chain Rule, we have

$$\frac{d}{dx}\left(e^{\cos\pi x}\right) = -\pi\sin\pi x\cdot e^{\cos\pi x}.$$

Therefore,

$$\frac{d}{dx}\left(e^{\cos \pi x}\right)\Big|_{x=1/2} = -\pi \underbrace{\sin \frac{\pi}{2}}_{1} \cdot \underbrace{e^{\cos(\pi/2)}}_{e^0 = 1} = -\pi.$$

Related Exercises 33, 35, 37 ◆

EXAMPLE 4 **Finding tangent lines**

a. Write an equation of the line tangent to the graph of $f(x) = 2x - \frac{e^x}{2}$ at the point $\left(0, -\frac{1}{2}\right)$.

b. Find the point(s) on the graph of f where the tangent line is horizontal.

SOLUTION »

a. To find the slope of the tangent line at $\left(0, -\frac{1}{2}\right)$, we first calculate $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x - \frac{e^x}{2}\right) \\ &= \frac{d}{dx}(2x) - \frac{d}{dx}\left(\frac{1}{2}e^x\right) && \text{Difference Rule} \\ &= 2 - \frac{1}{2}e^x. && \text{Evaluate derivatives.} \end{aligned}$$

It follows that the slope of the tangent line at $\left(0, -\frac{1}{2}\right)$ is

$$f'(0) = 2 - \frac{1}{2}e^0 = \frac{3}{2}.$$

Figure 7.20 shows the tangent line passing through $\left(0, -\frac{1}{2}\right)$; it has the equation

$$y - \left(-\frac{1}{2}\right) = \frac{3}{2}(x - 0) \quad \text{or} \quad y = \frac{3}{2}x - \frac{1}{2}.$$

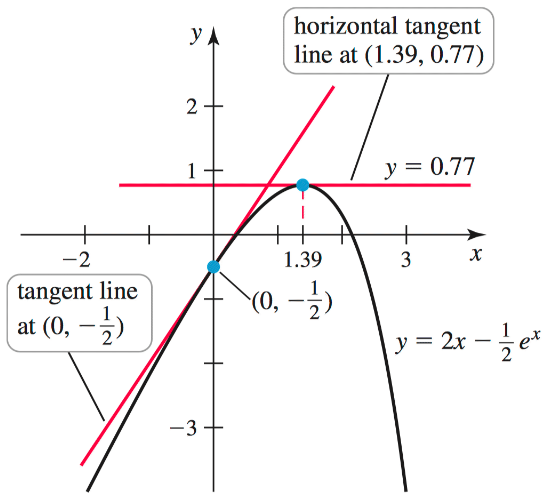


Figure 7.20

b. Because the slope of a horizontal tangent line is 0, our goal is to solve $f'(x) = 2 - \frac{1}{2}e^x = 0$. We multiply both sides of this equation by 2 and rearrange to arrive at the equation $e^x = 4$. Taking the natural logarithm of both sides, we find that $x = \ln 4$. Thus, $f'(x) = 0$ at $x = \ln 4 \approx 1.39$, and f has a horizontal tangent at $(\ln 4,\ f(\ln 4)) \approx (1.39,\ 0.77)$ (Figure 7.20).

Related Exercises 39–40 ◆

EXAMPLE 5 Integrals with e^x

Evaluate the following integrals.

a. $\int e^{10x}\,dx$

b. $\int \frac{e^x}{1+e^x}\,dx$

SOLUTION »

a. We let $u = 10x$, which implies $du = 10\,dx$, or $dx = \frac{1}{10}\,du$:

$$\begin{aligned}
\int \underbrace{e^{10x}}_{e^u}\,\underbrace{dx}_{\frac{1}{10}du} &= \int e^u \frac{1}{10}\,du && u = 10x,\ du = 10\,dx \\
&= \frac{1}{10}\int e^u\,du && \int c\,f(x)\,dx = c\int f(x)\,dx \\
&= \frac{1}{10}e^u + C && \text{Antiderivative} \\
&= \frac{1}{10}e^{10x} + C. && \text{Replace } u \text{ with } 10x.
\end{aligned}$$

The procedure used here can be generalized by replacing 10 with a nonzero constant a to obtain

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax} + C.$$

b. The change of variables $u = 1 + e^x$ implies $du = e^x\,dx$:

$$\begin{aligned}
\int \underbrace{\frac{1}{1+e^x}}_{u}\,\underbrace{e^x\,dx}_{du} &= \int \frac{1}{u}\,du && u = 1 + e^x,\ du = e^x\,dx \\
&= \ln|u| + C && \text{Antiderivative of } u^{-1} \\
&= \ln(1+e^x) + C. && \text{Replace } u \text{ by } 1 + e^x.
\end{aligned}$$

Note that the absolute value may be removed from $\ln|u|$ because $1 + e^x > 0$, for all x.

Related Exercises 52, 60 ◆

EXAMPLE 6 **Arc length of an exponential curve**

Find the length of the curve $f(x) = 2\,e^x + \frac{1}{8}\,e^{-x}$ on the interval $[0, \ln 2]$.

SOLUTION »

We first calculate $f'(x) = 2\,e^x - \frac{1}{8}\,e^{-x}$ and $f'(x)^2 = 4\,e^{2x} - \frac{1}{2} + \frac{1}{64}\,e^{-2x}$. The length of the curve on the interval $[0, \ln 2]$ is

$$\begin{aligned}
L = \int_0^{\ln 2} \sqrt{1 + f'(x)^2}\,dx &= \int_0^{\ln 2} \sqrt{1 + \left(4\,e^{2x} - \frac{1}{2} + \frac{1}{64}\,e^{-2x}\right)}\,dx \\
&= \int_0^{\ln 2} \sqrt{4\,e^{2x} + \frac{1}{2} + \frac{1}{64}\,e^{-2x}}\,dx && \text{Simplify.} \\
&= \int_0^{\ln 2} \sqrt{\left(2\,e^x + \frac{1}{8}\,e^{-x}\right)^2}\,dx && \text{Factor.} \\
&= \int_0^{\ln 2} \left(2\,e^x + \frac{1}{8}\,e^{-x}\right)dx && \text{Simplify.} \\
&= \left(2\,e^x - \frac{1}{8}\,e^{-x}\right)\Bigg|_0^{\ln 2} = \frac{33}{16}. && \text{Evaluate the integral.}
\end{aligned}$$

Related Exercises 61–62 ◆

Logarithmic Differentiation »

Products, quotients, and powers of functions are usually differentiated using the derivative rules of the same name (perhaps combined with the Chain Rule). There are times, however, when the direct computation of a derivative is tedious. Consider the function

$$f(x) = \frac{\left(x^3 - 1\right)^4 \sqrt{3x - 1}}{x^2 + 4}.$$

We would need the Quotient, Product, and Chain Rules just to compute $f'(x)$, and simplifying the result would require additional work. The properties of logarithms reviewed in this section are useful for differentiating such functions.

Note »

The properties of logarithms needed for logarithmic differentiation (where $x > 0$, $y > 0$, and z is any real number):

1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln (x^z) = z \ln x$

All three properties are used in Example 7.

EXAMPLE 7 Logarithmic differentiation

Let $f(x) = \frac{\left(x^2+1\right)^4 e^x}{x^2+4}$ and compute $f'(x)$.

SOLUTION »

We begin by taking the natural logarithm of both sides and simplifying the result:

Note »

$$\begin{aligned}\ln f(x) &= \ln\left(\frac{\left(x^2+1\right)^4 e^x}{x^2+4}\right) \\ &= \ln\left(x^2+1\right)^4 + \ln e^x - \ln\left(x^2+4\right) && \ln xy = \ln x + \ln y \\ &= 4\ln\left(x^2+1\right) + x - \ln\left(x^2+4\right). && \ln x^y = y \ln x;\ \ln e^x = x\end{aligned}$$

We now differentiate both sides using the Chain Rule; specifically, the derivative of the left side is $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$. Therefore,

$$\frac{f'(x)}{f(x)} = 4 \cdot \frac{1}{x^2+1} \cdot 2x + 1 - \frac{1}{x^2+4} \cdot 2x.$$

Solving for $f'(x)$, we have

$$f'(x) = f(x)\left(\frac{8x}{x^2+1} + 1 - \frac{2x}{x^2+4}\right).$$

Finally, we replace $f(x)$ with the original function:

$$f'(x) = \frac{\left(x^2+1\right)^4 e^x}{x^2+4}\left(\frac{8x}{x^2+1} - \frac{2x}{x^2+4} + 1\right)$$

Related Exercises 63, 67 ◆

Logarithmic differentiation also provides a method for finding derivatives of tower functions, which are functions of the form $g(x)^{h(x)}$. The derivative of $f(x) = x^x$ is computed as follows, assuming $x > 0$:

$$f(x) = x^x$$

$$\ln f(x) = \ln x^x = x \ln x \quad \text{Take logarithms of both sides; use properties.}$$

$$\frac{1}{f(x)} f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} \quad \text{Differentiate both sides.}$$

$$f'(x) = f(x)(\ln x + 1) \quad \text{Solve for } f'(x) \text{ and simplify.}$$

$$f'(x) = x^x(\ln x + 1). \quad \text{Replace } f(x) \text{ with } x^x.$$

Exercises »

Getting Started »

Practice Exercises »

17–38. Derivatives involving ln *x* *Find the following derivatives. Give the intervals on which the results are valid.*

17. $\frac{d}{dx}(\ln 7x)$

18. $\frac{d}{dx}\left(x^2 \ln x\right)$

19. $\frac{d}{dx}\left(\ln x^2\right)$

20. $\frac{d}{dx}\left(\ln 2x^8\right)$

21. $\frac{d}{dx}(\ln |\sin x|)$

22. $\frac{d}{dx}\left(\frac{\ln x^2}{x}\right)$

23. $\frac{d}{dx}\left(\ln\left(\frac{x+1}{x-1}\right)\right)$

24. $\frac{d}{dx}(e^x \ln x)$

25. $\frac{d}{dx}\left(\left(x^2+1\right)\ln x\right)$

26. $\frac{d}{dx}\left(\ln\left|x^2-1\right|\right)$

27. $\frac{d}{dx}(\ln(\ln x))$

28. $\dfrac{d}{dx}\left(\ln\left(\cos^2 x\right)\right)$

29. $\dfrac{d}{dx}\left(\dfrac{\ln x}{\ln x + 1}\right)$

30. $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right)$

31. $\dfrac{d}{dx}\left(\dfrac{e^x}{e^x + 1}\right)$

32. $\dfrac{d}{dx}\left(\dfrac{2\,e^x - 1}{2\,e^x + 1}\right)$

33. $\dfrac{d}{dx}\left(9\,e^{-x} - 5\,e^{2\,x} - 6\,e^x\right)$

34. $\dfrac{d}{dx}\left(x\,e^{-x} - e^{2\,x}\right)$

35. $\dfrac{d}{dx}\left(\dfrac{e^{2\,x}}{e^{-x} + 2}\right)$

36. $\dfrac{d}{dx}(\cot e^x)$

37. $\dfrac{d}{dx}\left(e^{\sin 2\,x}\right)\Big|_{x=\pi/4}$

38. $\dfrac{d}{dx}\left(\ln\left(e^{2\,x} + 1\right)\right)\Big|_{x=\ln 2}$

39–40. Equations of tangent lines *Find an equation of the line tangent to the following curves at the point* $(a,\ f(a))$.

39. $y = \dfrac{e^x}{4} - x;\ a = 0$

40. $y = 2\,e^x - 1;\ a = \ln 3$

41–60. Integrals *Evaluate the following integrals. Include absolute values only when needed.*

41. $\displaystyle\int \frac{3}{x - 10}\,dx$

42. $\displaystyle\int \frac{dx}{4\,x - 3}$

43. $\int \left(\frac{2}{x-4} - \frac{3}{2x+1} \right) dx$

44. $\int \frac{x^2}{2x^3+1}\, dx$

45. $\int_0^3 \frac{2x-1}{x+1}\, dx$

46. $\int \frac{\sec^2 x}{\tan x}\, dx$

47. $\int_3^4 \frac{dx}{2x \ln x \ln^3(\ln x)}$

48. $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\, dx$

49. $\int_{e^2}^{e^3} \frac{dx}{x \ln x \ln(\ln x)}$

50. $\int_0^1 \frac{y \ln^4 \left(y^2+1\right)}{y^2+1}\, dy$

51. $\int \frac{e^{2x} - e^{-2x}}{2}\, dx$

52. $\int 3 e^{-4t}\, dt$

53. $\int \left(e^{2x}+1\right) dx$

54. $\frac{1}{2} \int_0^{\ln 2} e^x\, dx$

55. $\int_0^{\ln 3} e^x \left(e^{3x} + e^{2x} + e^x\right) dx$

56. $\int \left(2 e^{-10z} + 3 e^{5z}\right) dz$

57. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

58. $\int \frac{e^{\sin x}}{\sec x}\, dx$

59. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

60. $\displaystyle\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}\,dz$

61–62. Arc length *Find the length of the following curves.*

61. $x = 2\,e^{\sqrt{2}\,y} + \dfrac{1}{16}\,e^{-\sqrt{2}\,y}$, for $0 \le y \le \dfrac{\ln 2}{\sqrt{2}}$

62. $y = \dfrac{1}{2}\,(e^{x} + e^{-x})$, for $-\ln 2 \le x \le \ln 2$

63–70. Logarithmic differentiation *Use logarithmic differentiation to evaluate $f\,'(x)$.*

63. $f(x) = \dfrac{(x+1)^{10}}{(2\,x-4)^{8}}$

64. $f(x) = x^{2}\cos x$

65. $f(x) = x^{\ln x}$

66. $f(x) = \dfrac{\tan^{10} x}{(5\,x+3)^{6}}$

67. $f(x) = \dfrac{(x+1)^{3/2}\,(x-4)^{5/2}}{(5\,x+3)^{2/3}}$

68. $f(x) = \dfrac{x^{8}\cos^{3} x}{\sqrt{x-1}}$

69. $f(x) = (\sin x)^{\tan x}$

70. $f(x) = \left(1 + \dfrac{1}{x}\right)^{2\,x}$

71. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample. Assume $x > 0$ and $y > 0$.

a. $\ln xy = \ln x + \ln y$.

b. $\ln 0 = 1$.

c. $\ln (x + y) = \ln x + \ln y$.

d. $\dfrac{d}{dx}\left(e^{3}\right) = e^{3}$.

e. $\dfrac{d}{dx}\,(e^{x}) = x\,e^{x-1}$.

f. $\dfrac{d^n}{dx^n}\left(e^{3x}\right) = 3^n e^{3x}$, for any integer $n \ge 1$.

g. The area between the curve $y = \dfrac{1}{x}$ and the x-axis on the interval $[1, e]$ is 1.

72. **Logarithmic properties** Use the integral definition of the natural logarithm to prove directly that $\ln \dfrac{x}{y} = \ln x - \ln y$.

73. **Looking ahead: Integrals of tan *x* and cot *x*** Use a change of variables to verify each integral.

a. $\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C$

b. $\int \cot x \, dx = \ln |\sin x| + C$

T **74.** **Behavior at the origin** Using calculus and accurate sketches, explain how the graphs of $f(x) = x^p \ln x$ differ as $x \to 0^+$ for $p = \dfrac{1}{2}, 1, 2$.

75. **Average value** What is the average value of $f(x) = \dfrac{1}{x}$ on the interval $[1, p]$ for $p > 1$? What is the average value of f as $p \to \infty$?

76–97. Miscellaneous derivatives and integrals *Compute the following derivatives using a method of your choice.*

76. $\dfrac{d}{dx}\left(x^{2x}\right)$

77. $\dfrac{d}{dx}\left(e^{-10x^2}\right)$

78. $\dfrac{d}{dx}\left(x^{\tan x}\right)$

79. $\dfrac{d}{dx}\left(\ln \sqrt{10x}\right)$

80. $\dfrac{d}{dx}(x^e + e^x)$

81. $\dfrac{d}{dx}\left(\dfrac{\left(x^2+1\right)(x-3)}{(x+2)^3}\right)$

82. $\dfrac{d}{dx}\left(\ln\left(\sec^4 x \tan^2 x\right)\right)$

83. $\dfrac{d}{dx}(\sin(\sin(e^x)))$

84. $\dfrac{d}{dx}\left(\sin^2\left(e^{3x+1}\right)\right)$

85. $\dfrac{d}{dx}\left(\dfrac{x\,e^x}{x+1}\right)$

86. $\dfrac{d}{dx}\left(\left(\dfrac{e^x}{x+1}\right)^8\right)$

87. $\int x^2\, e^{x^3}\, dx$

88. $\int_0^{\pi} e^{\sin x} \cos x\, dx$

89. $\int_1^{2e} \dfrac{e^{\ln x}}{x}\, dx$

90. $\int \dfrac{\sin(\ln x)}{4x}\, dx$

91. $\int_1^{e^2} \dfrac{\ln^5 x}{x}\, dx$

92. $\int \dfrac{\ln^2 x + 2\ln x - 1}{x}\, dx$

93. $\int \dfrac{x}{x-2}\, dx$ (*Hint:* Let $u = x - 2$.)

94. $\int_0^{\ln 4} \dfrac{e^x}{3 + 2e^x}\, dx$

95. $\int_0^{\pi/6} \dfrac{\sin 2y}{\sin^2 y + 2}\, dy$ (*Hint:* $\sin 2y = 2 \sin y \cos y$.)

96. $\int \dfrac{e^{2x}}{e^{2x} + 1}\, dx$

97. $\int \dfrac{dt}{t \ln t^2}$

98–99. First and second derivative analysis

a. *Find the critical points of f.*

b. *Use the First Derivative Test to locate the local maximum and minimum values.*

c. *Determine the intervals on which f is concave up or concave down. Identify any inflection points.*

98. $f(x) = x^4\, e^{-x}$

99. $f(x) = x\,e^{-x^2/2}$

100–101. Linear approximation

a. *Find the linear approximation L to the function f at the point a.*

b. *Graph f and L on the same set of axes.*

c. *Use the linear approximation to estimate the given quantity.*

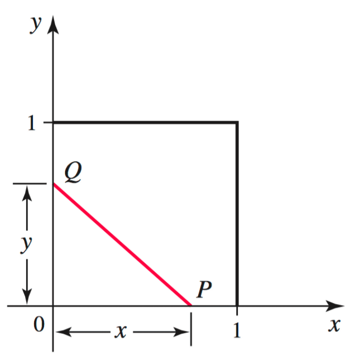
100. $f(x) = \ln(1+x)$; $a = 0$; $\ln 1.9$

101. $f(x) = e^{-x}$; $a = \ln 2$; $\dfrac{1}{e}$

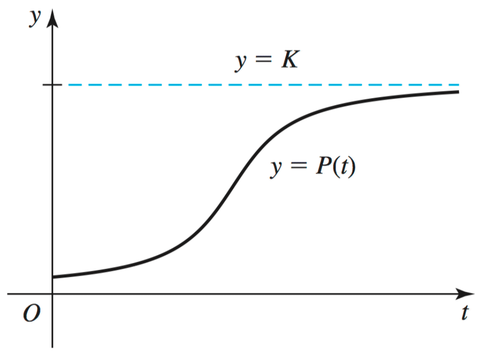
102. Solid of revolution The region bounded by the graphs of $x = 0$, $x = \sqrt{\ln y}$, and $x = \sqrt{2 - \ln y}$ in the first quadrant is revolved about the y-axis. What is the volume of the resulting solid?

Explorations and Challenges »

103. Probability as an integral Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square (see figure). What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \dfrac{1}{2}$ in order for the area condition to be met. Then argue that the required probability is $\dfrac{1}{2} + \displaystyle\int_{1/2}^{1} \frac{dx}{2x}$ and evaluate the integral.



104–107. Logistic growth *Scientists often use the* logistic growth *function* $P(t) = \dfrac{P_0 K}{P_0 + (K - P_0)\,e^{-r_0 t}}$ *to model population growth, where P_0 is the initial population at time $t = 0$, K is the* **carrying capacity**, *and r_0 is the* base growth rate. *The carrying capacity is a theoretical upper bound on the total population that the surrounding environment can support. The figure shows the sigmoid (S-shaped) curve associated with a typical logistic model.*



T **104. Population crash** The logistic model can be used for situations in which the initial population P_0 is above the carrying capacity K. For example, consider a deer population of 1500 on an island where a large fire has reduced the carrying capacity to 1000 deer.

a. Assuming a base growth rate of $r_0 = 0.1$ and an initial population of $P(0) = 1500$, write a logistic growth function for the deer population and graph it. Based on the graph, what happens to the deer population in the long run?

b. How fast (in deer per year) is the population declining immediately after the fire at $t = 0$?

c. How long does it take for the deer population to decline to 1200 deer?

T **105. Gone fishing** When a reservoir is created by a new dam, 50 fish are introduced into the reservoir, which has an estimated carrying capacity of 8000 fish. A logistic model of the fish population is $P(t) = \dfrac{400{,}000}{50 + 7950\, e^{-0.5\, t}}$, where t is measured in years.

a. Graph P using a graphing utility. Experiment with different windows until you produce an S-shaped curve characteristic of the logistic model. What window works well for this function?

b. How long does it take for the population to reach 5000 fish? How long does it take for the population to reach 90% of the carrying capacity?

c. How fast (in fish per year) is the population growing at $t = 0$? At $t = 5$?

d. Graph P' and use the graph to estimate the year in which the population is growing fastest.

T **106. World population (part 1)** The population of the world reached 6 billion in 1999 ($t = 0$). Assume the carrying capacity is 15 billion and the base growth rate is $r_0 = 0.025$ per year.

a. Write a logistic growth function for the world's population (in billions), and graph your equation on the interval $0 \le t \le 200$ using a graphing utility.

b. What will the population be in the year 2020? When will it reach 12 billion?

107. World population (part 2) The *relative growth rate r* of a function f measures the rate of change of the function compared to its value at a particular point. It is computed as $r(t) = \dfrac{f'(t)}{f(t)}$.

a. Confirm that the relative growth rate in 1999 ($t = 0$) for the logistic model in Exercise 72 is $r(0) = \dfrac{P'(0)}{P(0)} = 0.015$. This means the world's population was growing at 1.5% per year in 1999.

b. Compute the relative growth rate of the world's population in 2010 and 2020. What appears to be happening to the relative growth rate as time increases?

c. Evaluate $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \frac{P'(t)}{P(t)}$, where $P(t)$ is the logistic growth function from Exercise 106. What does your answer say about populations that follow a logistic growth pattern?

108. Snow plow problem With snow on the ground and falling at a constant rate, a snow plow began plowing down a long straight road at noon. The plow traveled twice as far in the first hour as it did in the second hour. At what time did the snow start falling? Assume the plowing rate is inversely proportional to the depth of the snow.

109. Depletion of natural resources Suppose that $r(t) = r_0 e^{-kt}$ is the rate at which a nation extracts oil, where $r_0 = 10^7$ barrels/yr is the current rate of extraction. Suppose also that the estimate of the total oil reserve is 2×10^9 barrels.

a. Find $Q(t)$, the total amount of oil extracted by the nation after t years.

b. Evaluate $\lim_{t\to\infty} Q(t)$ and explain the meaning of this limit.

c. Find the minimum decay constant k for which the total oil reserves will last forever.

d. Suppose $r_0 = 2\times10^7$ barrels/yr and the decay constant k is the minimum value found in part (c). How long will the total oil reserves last?

110. Derivative of ln |*x*| Differentiate $\ln x$ for $x > 0$ and differentiate $\ln(-x)$ for $x < 0$ to conclude that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$.

111. Properties of e^x Use the inverse relations between $\ln x$ and e^x $(\exp(x))$ and the properties of $\ln x$ to prove the following properties.

a. $\exp(0) = 1$

b. $\exp(x-y) = \frac{\exp(x)}{\exp(y)}$

c. $(\exp(x))^p = \exp(px)$, where p is an integer

112. ln *x* is unbounded Use the following argument to show that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to0^+} \ln x = -\infty$.

a. Make a sketch of the function $f(x) = \frac{1}{x}$ on the interval $[1, 2]$. Explain why the area of the region bounded by $y = f(x)$ and the x-axis on $[1, 2]$ is $\ln 2$.

b. Construct a rectangle over the interval $[1, 2]$ with height $\frac{1}{2}$. Explain why $\ln 2 > \frac{1}{2}$.

c. Show that $\ln 2^n > \frac{n}{2}$ and $\ln 2^{-n} < -\frac{n}{2}$.

d. Conclude that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to0^+} \ln x = -\infty$.

113. Bounds on *e* Use a left Riemann sum with $n = 2$ subintervals of equal length to approximate $\ln 2 = \int_1^2 \frac{dt}{t}$ and show that $\ln 2 < 1$. Use a right Riemann sum with $n = 7$ subintervals of equal length to approximate $\ln 3 = \int_1^3 \frac{dt}{t}$ and show that $\ln 3 > 1$.

114. Alternate proof of product property Assume that $y > 0$ is fixed and that $x > 0$. Show that $\frac{d}{dx}(\ln xy) = \frac{d}{dx}(\ln x)$. Recall that if two functions have the same derivative, they differ by a constant. Set $x = 1$ to evaluate the constant and prove that $\ln xy = \ln x + \ln y$.

115. Harmonic sum In Chapter 10, we will encounter the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Use a right Riemann sum to approximate $\int_1^n \frac{dx}{x}$ (with unit spacing between the grid points) to show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$. Use this fact to conclude that $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$ does not exist.

T **116. Tangency question** It is easily verified that the graphs of $y = x^2$ and $y = e^x$ have no points of intersection (for $x > 0$), while the graphs of $y = x^3$ and $y = e^x$ have two points of intersection. It follows that for some real number $2 < p < 3$, the graphs of $y = x^p$ and $y = e^x$ have exactly one point of intersection (for $x > 0$). Using analytical and/or graphical methods, determine p and the coordinates of the single point of intersection.

117. Property of exponents Prove that for real numbers x, y, a, and $b \neq 0$, $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.