7 Logarithmic and Exponential Functions

Chapter Preview In Chapter 1, we presented a catalog of elementary functions that included polynomial, algebraic, and trigonometric functions, exploring their applications along the way. Until now, we have studied the calculus of these three families of functions, exploring their applications along the way. However, many areas of mathematics rely on other functions, such as exponential and logarithm functions, which are known to you from algebra. It turns out that an exponential function of the form b^x, where $b > 0$, is the *inverse function* of the logarithmic function $\log_b x$ (and vice versa). In light of this fact, the chapter opens with a general discussion of inverse functions. We then define the *natural logarithmic function* and its inverse the *natural exponential function*, and develop the various derivatives and integrals associated with these functions. Inverse functions are next put to work to define the inverse trigonometric functions and to produce their derivative and integral properties. With exponential and logarithmic functions in the picture, we can resume the discussion of l'Hôpital's Rule that began in Chapter 4. New indeterminate forms are explored, leading to a ranking of functions by their growth rates. Finally, the chapter concludes with a study of *hyperbolic functions* and their inverses; these functions are related to exponential, logarithmic and trigonometric functions. Throughout the chapter, we emphasize the many practical applications of each of these families of functions.

7.1 Inverse Functions

From your study of algebra, you know that when a function has an *inverse function*, the two functions are related in special ways. Roughly speaking, the action of one function undoes the action of the other. In this section, we begin with a review of inverse functions: when they exist, how to find them, and how to graph them. We then investigate the relationship between the derivative of a function and the derivative of its inverse function. With this background, we devote much of the rest of the chapter to developing new functions that arise as the inverses of familiar functions.

Existence of Inverse Functions »

Consider the linear function $f(x) = 2x$, which takes any value of x and doubles it. The function that reverses this process by taking any value of $f(x) = 2x$ and mapping it back to x is called the *inverse function* of f, denoted f^{-1}. In this case, the inverse function is $f^{-1}(x) = \frac{x}{2}$. The effect of applying these two functions in succession looks like this:

$$x \xrightarrow{f} 2x \xrightarrow{f^{-1}} x.$$

We now generalize this idea.

Note »

The notation f^{-1} for the inverse can be confusing. The inverse is not the reciprocal; that is, $f^{-1}(x)$ is not $\frac{1}{f(x)} = (f(x))^{-1}$. We adopt the common convention of using simply *inverse* to mean *inverse function*.

DEFINITION **Inverse Function**

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$ (**Figure 7.1**).

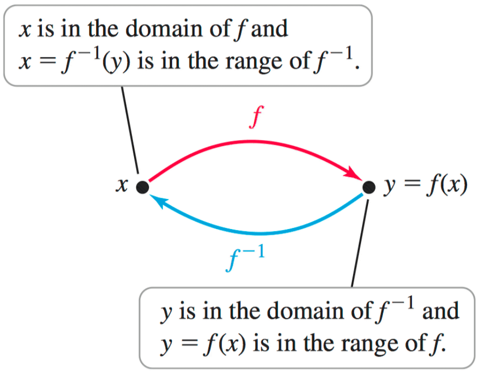


Figure 7.1

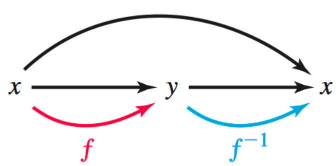
Quick Check 1 What is the inverse of $f(x) = \frac{1}{3}x$? What is the inverse of $f(x) = x - 7$? ◆

Answer »

$f^{-1}(x) = 3x$; $f^{-1}(x) = x + 7$

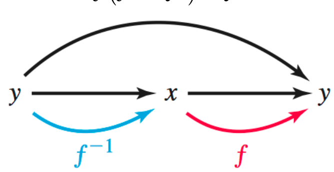
Because the inverse "undoes" the original function, if we start with a value of x, apply f to it, and then apply f^{-1} to the result, we recover the original value of x; that is,

$$f^{-1}(f(x)) = x.$$



Similarly, if we apply f^{-1} to a value of y and then apply f to the result, we recover the original value of y; that is,

$$f\left(f^{-1}(y)\right) = y.$$



One-to-One Functions

We have defined the inverse of a function, but said nothing about when it exists. To ensure that f has an inverse function on a domain, f must be *one-to-one* on that domain. This property means that every output of the function f corresponds to exactly one input. The one-to-one property is checked graphically by using the *horizontal line test*.

Note »

> The vertical line test determines whether f is a function. The horizontal line test determines whether f is one-to-one.

DEFINITION **One-to-One Functions and the Horizontal Line Test**

A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D. The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once (**Figure 7.2**).

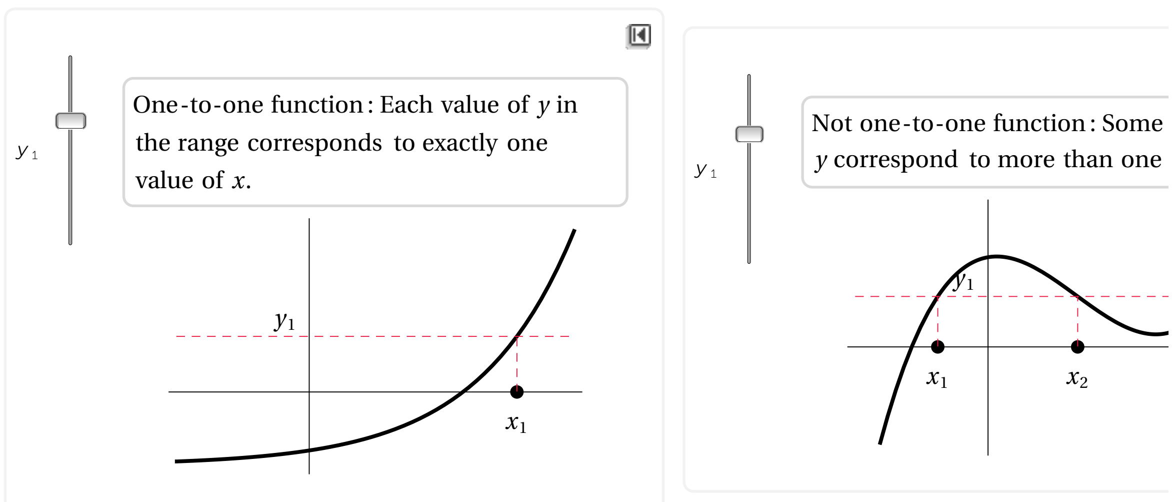


Figure 7.2

For example, in **Figure 7.3**, some horizontal lines intersect the graph of $f(x) = x^2$ twice. Therefore, f does not have an inverse function on the interval $(-\infty, \infty)$. However, if f is restricted to the interval $(-\infty, 0]$ or $[0, \infty)$, then it does pass the horizontal line test, and it is one-to-one on these intervals.

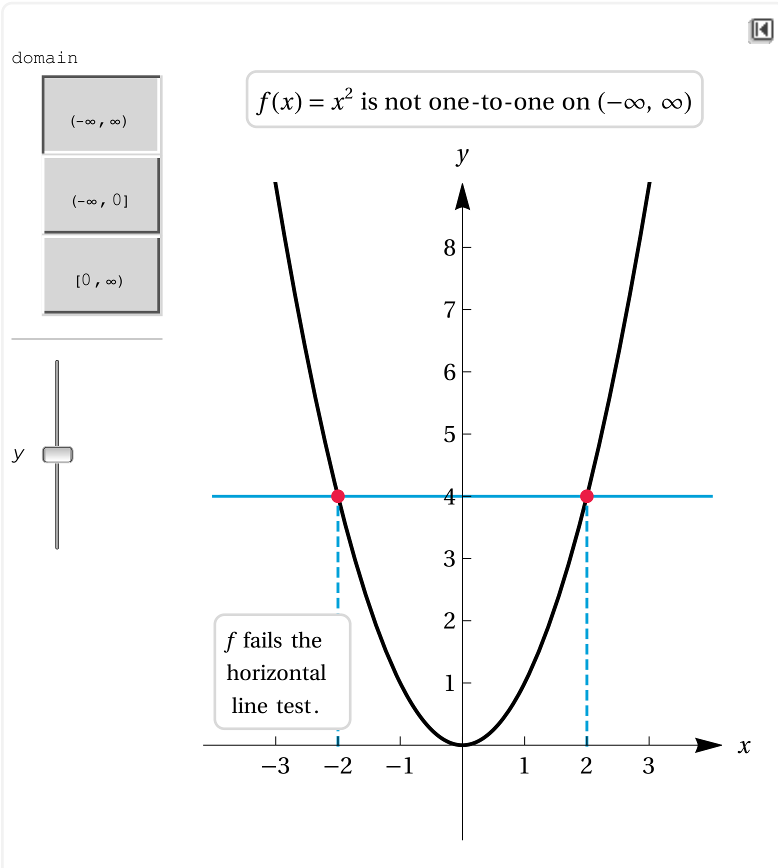


Figure 7.3

EXAMPLE 1 One-to-one functions

Determine the (largest possible) intervals on which the function $f(x) = 2x^2 - x^4$ (**Figure 7.4**) is one-to-one.

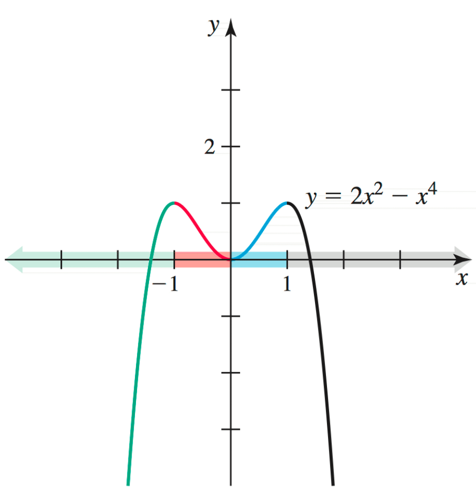


Figure 7.4

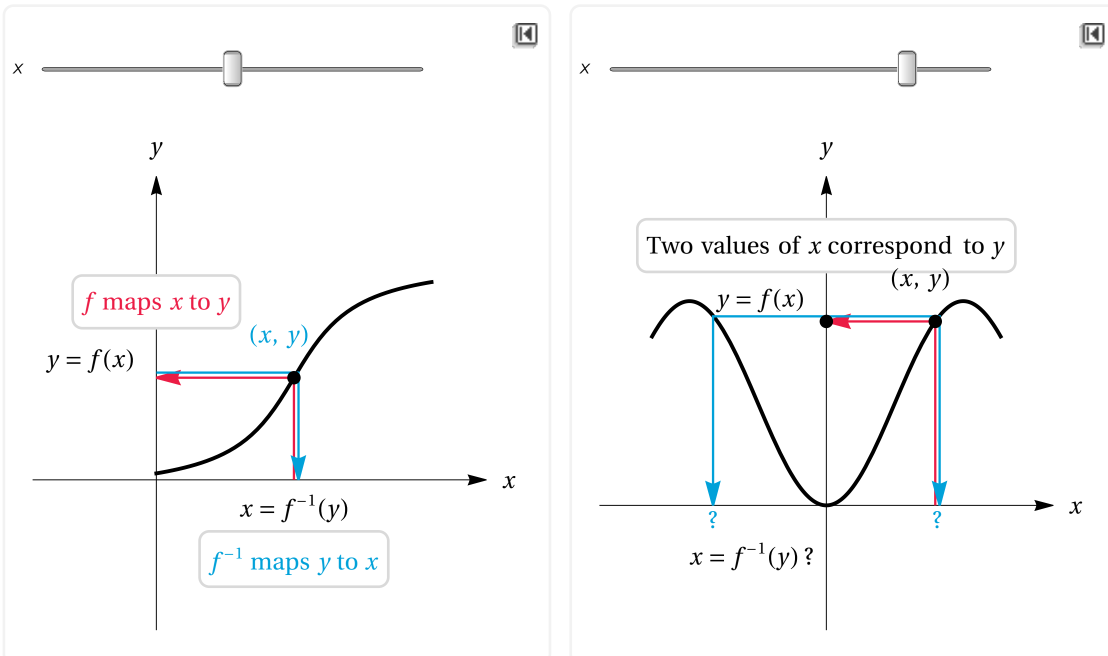
SOLUTION »

The function is not one-to-one on the entire real line because it fails the horizontal line test. However, on the intervals $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, \infty)$, f is one-to-one. The function is also one-to-one on any subinterval of these four intervals.

Related Exercises 5–6 ◆

Conditions for the Existence of Inverse Functions

Figure 7.5a illustrates the actions of a one-to-one function f and its inverse f^{-1}. We see that f maps a value of x to a unique value of y. In turn, f^{-1} maps that value of y back to the original value of x. When f is *not* one-to-one, this procedure cannot be carried out (**Figure 7.5b**).



(a) (b)

Figure 7.5

THEOREM 7.1 Existence of Inverse Functions

Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f\left(f^{-1}(y)\right) = y,$$

where x is in D and y is in R.

Note »

The statement that a one-to-one function has an inverse may be plausible based on its graph. However, the proof of this theorem is fairly technical and is omitted.

Quick Check 2 The function that gives degrees Fahrenheit in terms of degrees Celsius is $F = \frac{9}{5}C + 32$. Why does this function have an inverse? ◆

Answer »

For every Fahrenheit temperature, there is exactly one Celsius temperature, and vice versa. The given relation is also a linear function. It is one-to-one, so it has an inverse function.

EXAMPLE 2 Does an inverse exist?

Determine the largest intervals on which $f(x) = x^2 - 1$ has an inverse function.

SOLUTION »

On the interval $(-\infty, \infty)$ the function does not pass the horizontal line test and is not one-to-one (**Figure 7.6**). However, if f is restricted to the intervals $(-\infty, 0]$ or $[0, \infty)$, then it is one-to-one and an inverse exists.

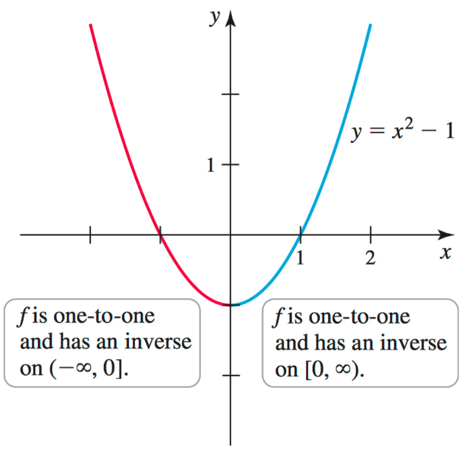


Figure 7.6

Related Exercises 20, 22 ◆

Finding Inverse Functions

The crux of finding an inverse for a function f is solving the equation $y = f(x)$ for x in terms of y. If it is possible to do so, then we have found a relationship of the form $x = f^{-1}(y)$. Interchanging x and y in $x = f^{-1}(y)$ so that x is the independent variable (which is the customary role for x), the inverse has the form $y = f^{-1}(x)$. Notice that if f is not one-to-one, this process leads to more than one inverse function.

PROCEDURE Finding an Inverse

Suppose f is one-to-one on an interval I. To find f^{-1}:

1. Solve $y = f(x)$ for x. If necessary, restrict the resulting function so that x lies in I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Note »

Once you find a formula for f^{-1} you can check your work by verifying that $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$.

EXAMPLE 3 **Finding inverse functions**

Find the inverse(s) of the following functions. Restrict the domain of f if necessary.

a. $f(x) = 2x + 6$

b. $f(x) = x^2 - 1$

SOLUTION »

a. Linear functions (except for constant linear functions) are one-to-one on the entire real line. Therefore, an inverse function for f exists for all values of x.

Note »

Step 1: Solve $y = f(x)$ for x: We see that $y = 2x + 6$ implies that $2x = y - 6$, or $x = \frac{1}{2}y - 3$.

Step 2: Interchange x and y and write $y = f^{-1}(x)$:

$$y = f^{-1}(x) = \frac{1}{2}x - 3.$$

It is instructive to verify that the inverse relations $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$ are satisfied:

$$f\left(f^{-1}(x)\right) = f\left(\frac{1}{2}x - 3\right) = \underbrace{2\left(\frac{1}{2}x - 3\right) + 6}_{f(x)=2x+6} = x - 6 + 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(2x + 6) = \underbrace{\frac{1}{2}(2x + 6) - 3}_{f^{-1}(x)=\frac{1}{2}x-3} = x + 3 - 3 = x.$$

b. As shown in Example 2, the function $f(x) = x^2 - 1$ is not one-to-one on the entire real line; however, it is one-to-one on $(-\infty, 0]$ and on $[0, \infty)$ (**Figure 7.7a**). If we restrict our attention to either of these intervals, then an inverse function can be found.

Step 1: Solve $y = f(x)$ for x:

$$y = x^2 - 1$$
$$x^2 = y + 1$$
$$x = \begin{cases} \sqrt{y+1} \\ -\sqrt{y+1}. \end{cases}$$

Each branch of the square root corresponds to an inverse function.

Step 2: Interchange x and y and write $y = f^{-1}(x)$:

$$y = f^{-1}(x) = \sqrt{x+1} \quad \text{or} \quad y = f^{-1}(x) = -\sqrt{x+1}.$$

The interpretation of this result is important. Taking the positive branch of the square root, the inverse function $y = f^{-1}(x) = \sqrt{x+1}$ gives positive values of y; it corresponds to the branch of $f(x) = x^2 - 1$ on the interval $[0, \infty)$

(**Figure 7.7b**). The negative branch of the square root, $y = f^{-1}(x) = -\sqrt{x+1}$, is another inverse function that gives negative values of y; it corresponds to the branch of $f(x) = x^2 - 1$ on the interval $(-\infty, 0]$.

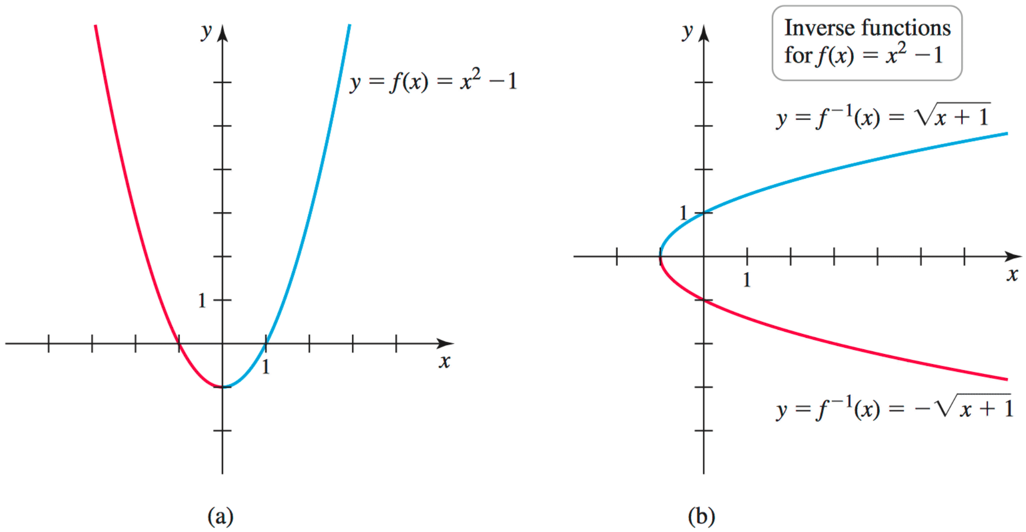


Figure 7.7

Related Exercises 31–32 ◆

Quick Check 3 On what interval(s) does the function $f(x) = x^3$ have an inverse? ◆

Answer »

> The function $f(x) = x^3$ is one-to-one on $(-\infty, \infty)$, so it has an inverse for all values of x.

Graphing Inverse Functions »

The graphs of a function and its inverse have a special relationship, which is illustrated in the following example.

EXAMPLE 4 Graphing inverse functions

Plot f and f^{-1} on the same coordinate axes.

a. $f(x) = 2x + 6$

b. $f(x) = \sqrt{x-1}$

SOLUTION »

a. The inverse of $f(x) = 2x + 6$, found in Example 3, is

$$y = f^{-1}(x) = \frac{x}{2} - 3.$$

The graphs of f and f^{-1} are shown in **Figure 7.8** Notice that both f and f^{-1} are increasing linear functions and they intersect at $(-6, -6)$.

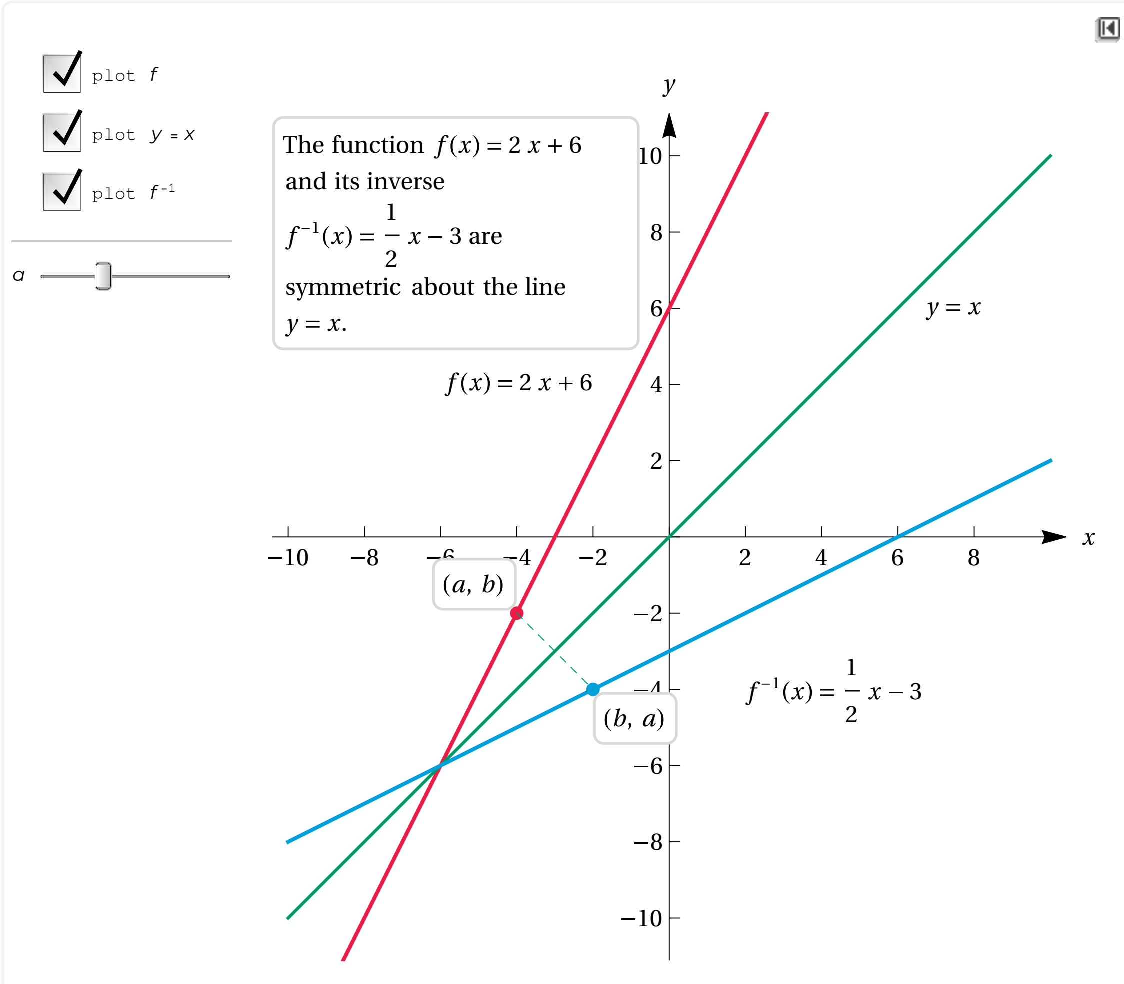


Figure 7.8

b. The domain of $f(x) = \sqrt{x-1}$ is $[1, \infty)$ and its range is $[0, \infty)$. On this domain, f is one-to-one and has an inverse. It can be found in two steps:

Step 1: Solve $y = \sqrt{x-1}$ for x:

$$y^2 = x - 1 \quad \text{or} \quad x = y^2 + 1.$$

Step 2: Interchange x and y and write $y = f^{-1}(x)$:

$$y = f^{-1}(x) = x^2 + 1.$$

The graphs of f and f^{-1} are shown in **Figure 7.9**; notice that the domain of f^{-1} (which is $x \geq 0$) corresponds to the range of f (which is $y \geq 0$).

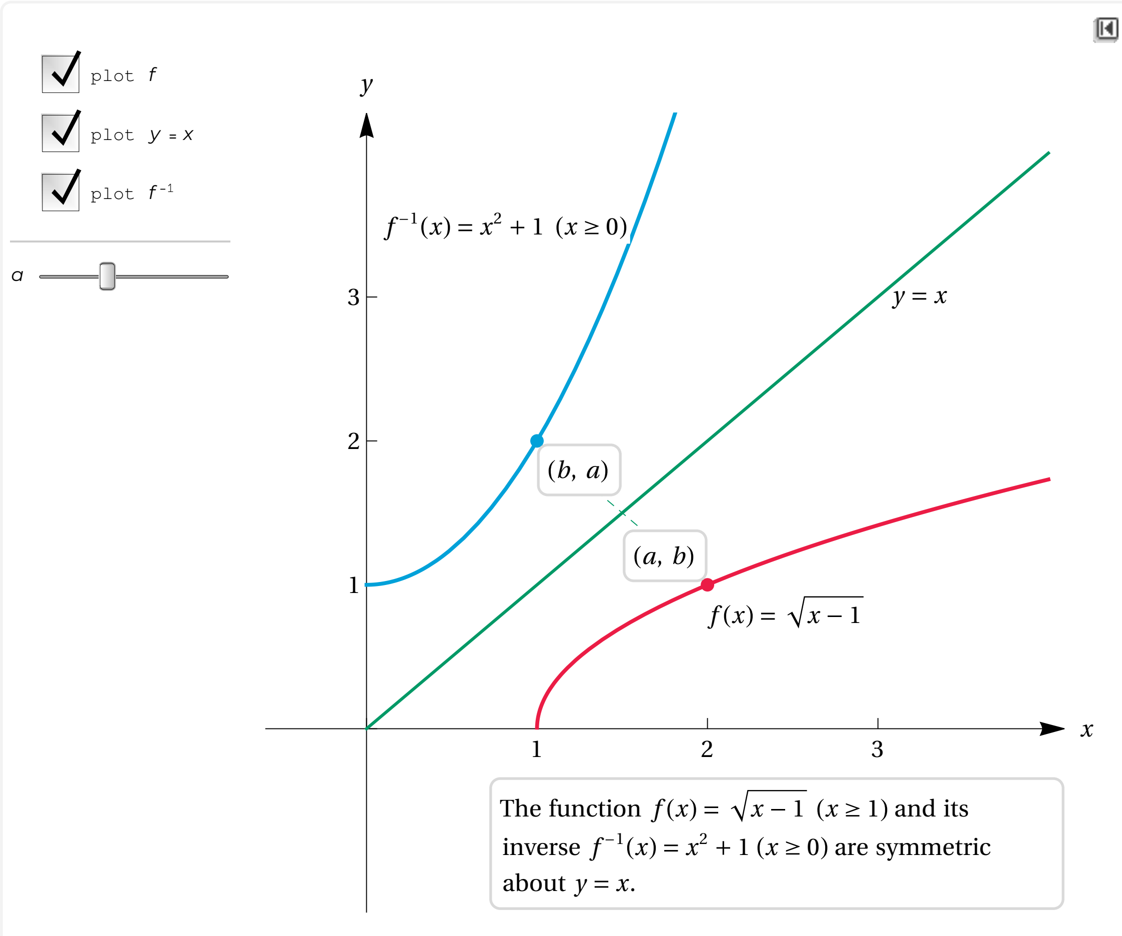


Figure 7.9

Related Exercises 24–25 ◆

Looking closely at the graphs in Figure 7.8 and Figure 7.9, you see a symmetry that always occurs when a function and its inverse are plotted on the same set of axes. In each figure, one curve is the reflection of the other curve across the line $y = x$. These curves are *symmetric about the line* $y = x$, which means that the point (a, b) is on one curve whenever the point (b, a) is on the other curve (**Figure 7.10**).

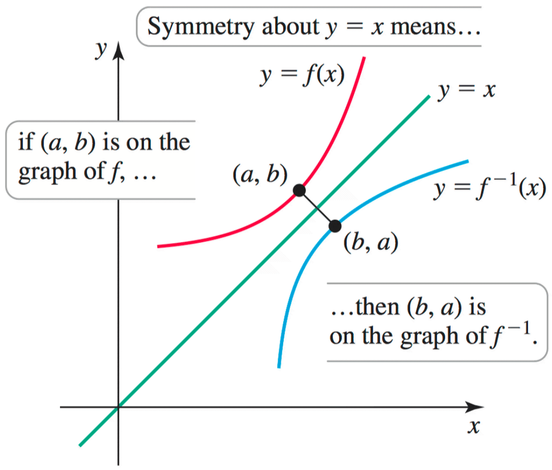


Figure 7.10

The explanation for the symmetry comes directly from the definition of the inverse. Suppose that the point $(a,\ b)$ is on the graph of $y = f(x)$, which means that $b = f(a)$. By the definition of the inverse function, we know that $a = f^{-1}(b)$, which means that the point $(b,\ a)$ is on the graph of $y = f^{-1}(x)$. This argument applies to all relevant points $(a,\ b)$, so whenever $(a,\ b)$ is on the graph of f, $(b,\ a)$ is on the graph of f^{-1}. As a consequence, the graphs are symmetric about the line $y = x$.

Now suppose a function f is continuous and one-to-one on an interval I. Reflecting the graph of f through the line $y = x$ generates the graph of f^{-1}. The reflection process introduces no discontinuities in the graph of f^{-1}, so it is plausible (and indeed, true) that f^{-1} is continuous on the interval corresponding to I. We state this fact without a formal proof.

THEOREM 7.2 Continuity of Inverse Functions

If a continuous function f has an inverse on an interval I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Derivatives of Inverse Functions »

Here is an important question that bears on upcoming work: Given a function f that is one-to-one on an interval and its derivative $f\,'$, how do we evaluate the derivative of f^{-1}? The key to finding the derivative of the inverse function lies in the symmetry of the graphs of f and f^{-1}.

EXAMPLE 5 Linear functions, inverses, and derivatives

Consider the general linear function $y = f(x) = m\,x + b$, where $m \neq 0$ and b are constants.

a. Write the inverse of f in the form $y = f^{-1}(x)$.

b. Find the derivative of the inverse $\dfrac{d}{dx}\left(f^{-1}(x)\right)$.

c. Consider the specific case $f(x) = 2\,x - 12$. Graph f and f^{-1}, and find the slope of each line.

SOLUTION »

a. Solving $y = mx + b$ for x, we find that $mx = y - b$, or

$$x = \frac{y}{m} - \frac{b}{m}.$$

Writing this function in the form $y = f^{-1}(x)$ (by reversing the roles of x and y), we have

$$y = f^{-1}(x) = \frac{x}{m} - \frac{b}{m},$$

which describes a line with slope $\frac{1}{m}$.

b. The derivative of f^{-1} is

$$\left(f^{-1}\right)'(x) = \frac{1}{m}.$$

Notice that $f'(x) = m$, so the derivative of f^{-1} is the reciprocal of f'.

c. In the case that $f(x) = 2x - 12$, we have $f^{-1}(x) = \frac{x}{2} + 6$. The graphs of these two lines are symmetric about the line $y = x$ (**Figure 7.11**). Furthermore, the slope of the line $y = f(x)$ is 2 and the slope of $y = f^{-1}(x)$ is $\frac{1}{2}$; that is, the slopes (and therefore the derivatives) are reciprocals of each other.

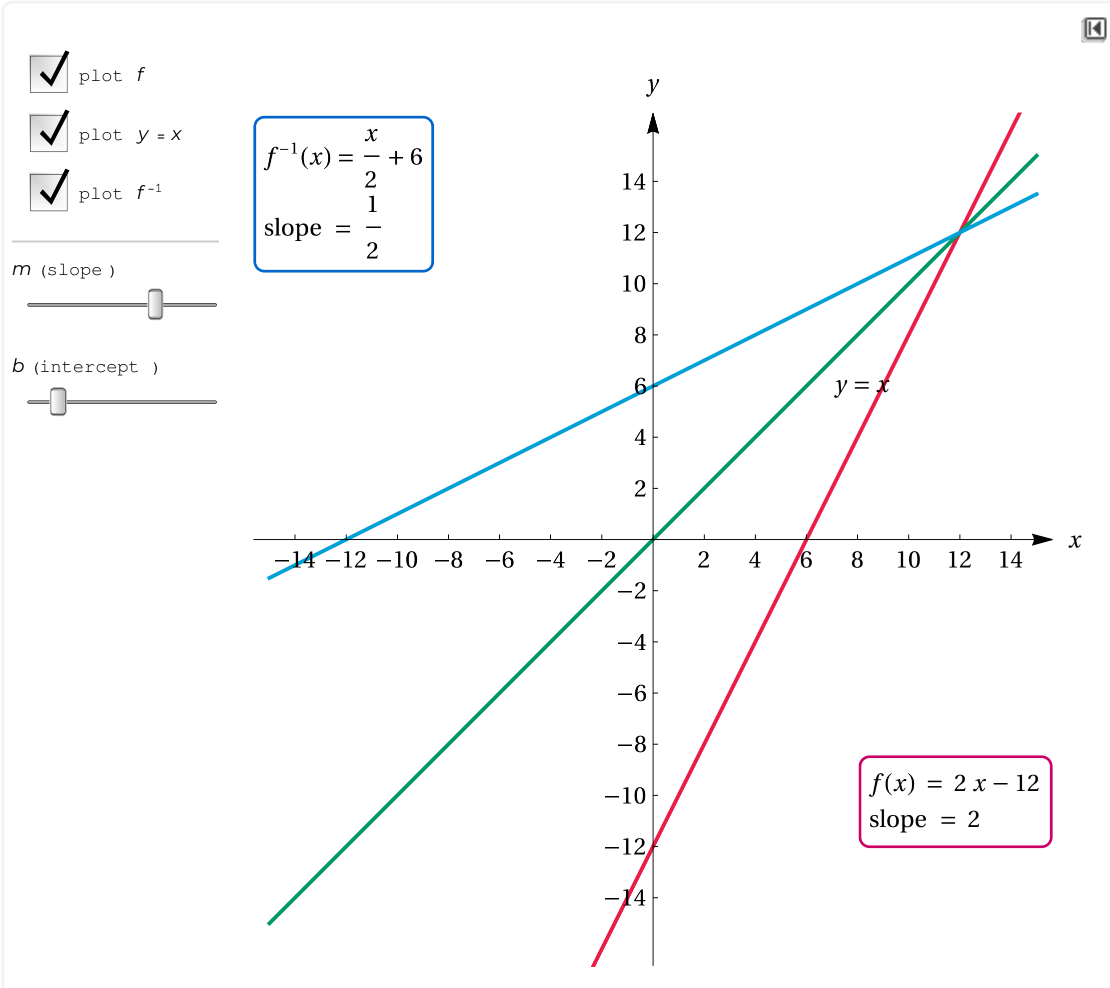


Figure 7.11

Related Exercises 39–40 ◆

The reciprocal property obeyed by f' and $(f^{-1})'$ in Example 5 holds for all functions. **Figure 7.12** shows the graphs of a typical one-to-one function and its inverse. It also shows a pair of symmetric points—(x_0, y_0) on the graph f and (y_0, x_0) on the graph of f^{-1}—along with the tangent lines at these points. Notice that as the lines tangent to the graph of f get steeper (as x increases), the corresponding lines tangent to the graph of f^{-1} get less steep. The next theorem makes this relationship precise.

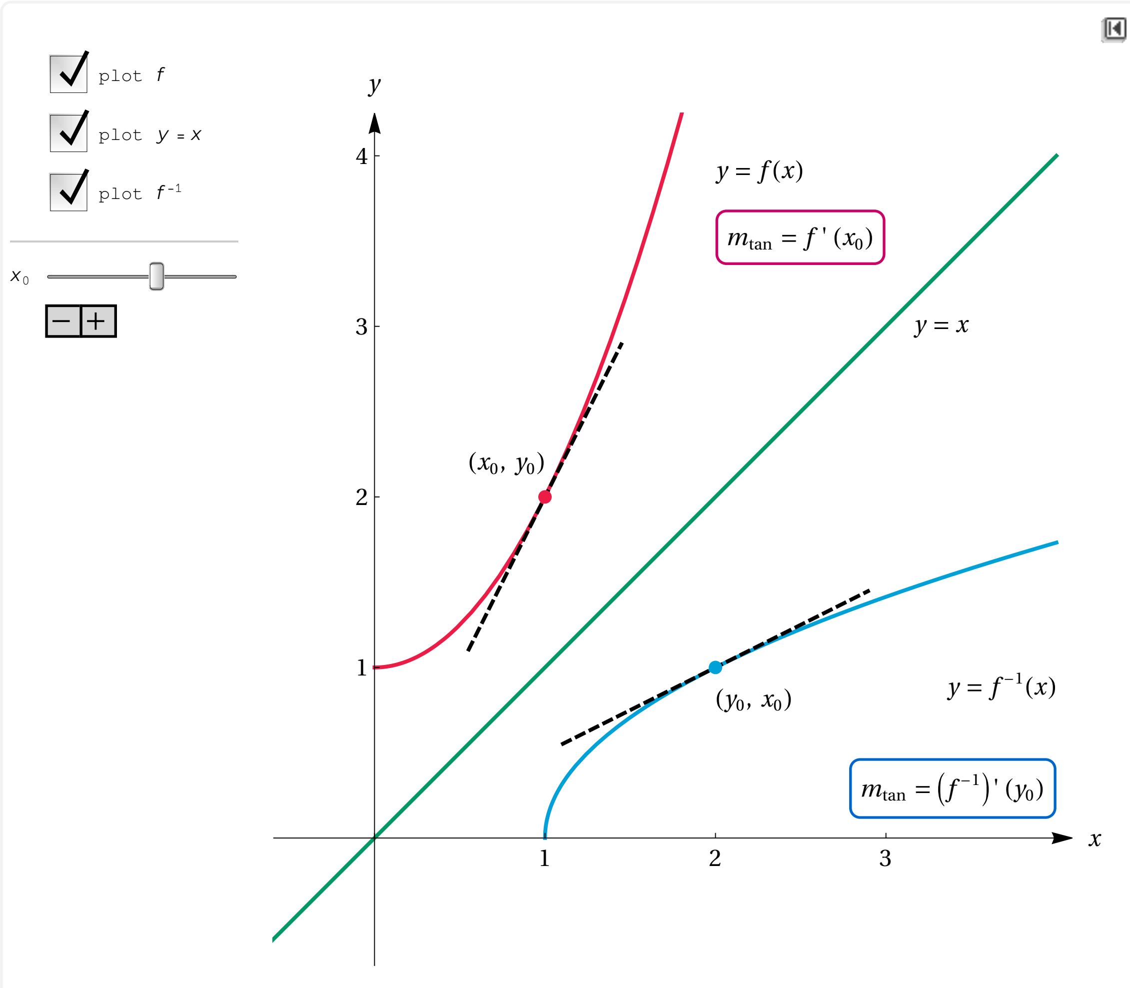


Figure 7.12

THEOREM 7.3 Derivative of the Inverse Function

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}, \quad \text{where } y_0 = f(x_0).$$

Note »

The result of Theorem 7.3 is also written in the form

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

or

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

To understand this theorem, suppose that (x_0, y_0) is a point on the graph of f, which means that (y_0, x_0) is the corresponding point on the graph of f^{-1}. Then the slope of the line tangent to the graph of f^{-1} at the point

(y_0, x_0) is the reciprocal of the slope of the line tangent to the graph of f at the point (x_0, y_0). Importantly, the theorem says that we can evaluate the derivative of the inverse function without finding the inverse function itself.

Proof: Before doing a short calculation, we note two facts:

- At a point x_0 where f is differentiable, $y_0 = f(x_0)$ and $x_0 = f^{-1}(y_0)$.
- Because f is continuous at x_0 (Theorem 3.1), which implies that f^{-1} is also continuous at y_0 (Theorem 7.2). Therefore, as $y \to y_0$, $x \to x_0$.

Using the definition of the derivative, we have

$$\begin{aligned}
\left(f^{-1}\right)'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} && \text{Definition of derivative of } f^{-1} \\
&= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} && y = f(x) \text{ and } x = f^{-1}(y);\ x \to x_0 \text{ as } y \to y_0 \\
&= \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} && \frac{a}{b} = \frac{1}{b/a} \\
&= \frac{1}{f'(x_0)}. && \text{Definition of derivative of } f
\end{aligned}$$

We have shown that $\left(f^{-1}\right)'(y_0)$ exists (f^{-1} is differentiable at y_0) and it equals the reciprocal of $f'(x_0)$. ◆

Quick Check 4 Sketch the graphs of $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$. Then verify that Theorem 7.3 holds at the point $(1, 1)$. ◆

Answer »

$f'(1) = 3, \left(f^{-1}\right)'(1) = \dfrac{1}{3}$.

EXAMPLE 6 Derivative of an inverse function

The function $f(x) = \sqrt{x} + x^2 + 1$ is one-to-one for $x \geq 0$, and has an inverse on that interval. Find the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$.

SOLUTION »

The point $(1, 3)$ is on the graph of f; therefore, $(3, 1)$ is on the graph of f^{-1}. In this case, the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$ is the reciprocal of the slope of the curve $y = f(x)$ at $(1, 3)$ (**Figure 7.13**). Note that $f'(x) = \dfrac{1}{2\sqrt{x}} + 2x$, which means that $f'(1) = \dfrac{1}{2} + 2 = \dfrac{5}{2}$. Therefore,

$$\left(f^{-1}\right)'(3) = \frac{1}{f'(1)} = \frac{1}{5/2} = \frac{2}{5}.$$

Observe that it is not necessary to find a formula for f^{-1} in order to evaluate its derivative at a point.

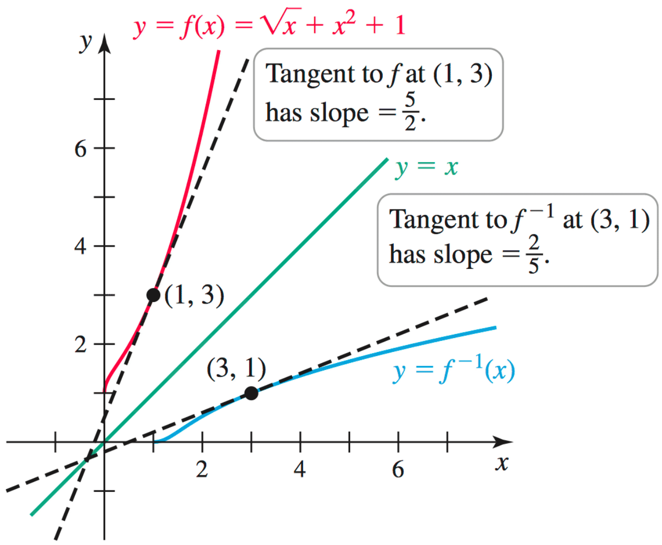


Figure 7.13

Related Exercises 47–48 ◆

EXAMPLE 7 Derivatives of an inverse function

Use the values of a one-to-one differentiable function in Table 7.1 to compute the indicated derivatives or state that the derivative cannot be determined.

Table 7.1

x	-1	0	1	2	3
$f(x)$	2	3	5	6	7
$f'(x)$	$\frac{1}{2}$	2	$\frac{3}{2}$	1	$\frac{2}{3}$

a. $\left(f^{-1}\right)'(5)$

b. $\left(f^{-1}\right)'(2)$

c. $\left(f^{-1}\right)'(1)$

SOLUTION »

We use the relationship $\left(f^{-1}\right)'(y_0) = \dfrac{1}{f'(x_0)}$, where $y_0 = f(x_0)$.

a. In this case, $y_0 = f(x_0) = 5$. Using the Table 7.1, we see that $x_0 = 1$ and $f'(1) = \dfrac{3}{2}$. Therefore,

$\left(f^{-1}\right)'(5) = \dfrac{1}{f'(1)} = \dfrac{2}{3}$.

b. In this case, $y_0 = f(x_0) = 2$, which implies that $x_0 = -1$ and $f'(-1) = \dfrac{1}{2}$. Therefore, $\left(f^{-1}\right)'(2) = \dfrac{1}{f'(-1)} = 2$.

c. With $y_0 = f(x_0) = 1$, Table 7.1 does not supply a value of x_0. Therefore, $f'(x_0)$ cannot be determined.

Related Exercises 53–54 ◆

Exercises »

Getting Started »

Practice Exercises »

17–22. Where do inverses exist? *Use analytical and/or graphical methods to determine the largest possible sets of points on which the following functions have an inverse.*

17. $f(x) = 3x + 4$

18. $f(x) = |2x + 1|$

19. $f(x) = \dfrac{1}{x - 5}$

20. $f(x) = -(6 - x)^2$

21. $f(x) = \dfrac{1}{x^2}$

22. $f(x) = x^2 - 2x + 8$ (*Hint:* Complete the square.)

23–28. Graphing inverse functions *Find the inverse function (on the given interval, if specified) and graph both f and f^{-1} on the same set of axes. Check your work by looking for the required symmetry in the graphs.*

23. $f(x) = 8 - 4x$

24. $f(x) = 3x + 5$

25. $f(x) = \sqrt{x + 2}$

26. $f(x) = \sqrt{3 - x}$

27. $f(x) = (x - 2)^2 - 1$, for $x \geq 2$

28. $f(x) = x^2 + 4$, for $x \geq 0$

29–36. Finding inverse functions

- ***a.*** *Find the inverse of each function (on the given interval, if specified) and write it in the form $y = f^{-1}(x)$.*
- ***b.*** *Verify the relationships $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$.*

29. $f(x) = 6 - 4x$

30. $f(x) = 3x^3$

31. $f(x) = (x - 7)^{5/3}$

32. $f(x) = x^2 - 16$, for $x \le 0$

33. $f(x) = 1 - \dfrac{1}{x}$

34. $f(x) = \dfrac{2}{x^2 + 1}$, for $x \ge 0$

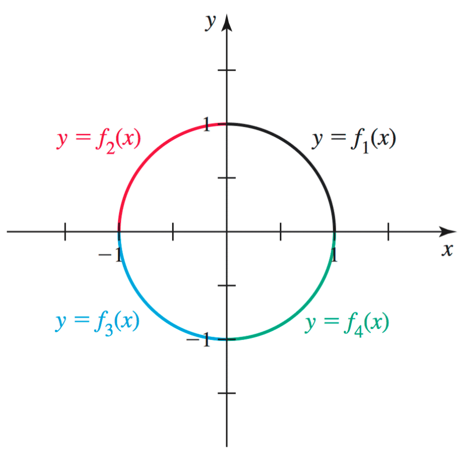
35. $f(x) = \dfrac{6}{x^2 - 9}$, for $x > 3$

36. $f(x) = \dfrac{x}{x - 2}$, for $x > 2$

37. **Splitting up curves** The unit circle $x^2 + y^2 = 1$ consists of four one-to-one functions, $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ (see figure).

a. Find the domain and a formula for each function.

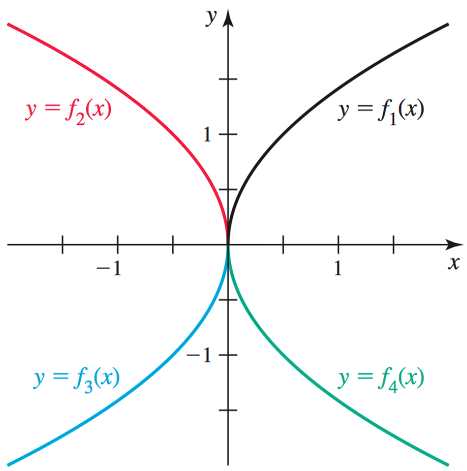
b. Find the inverse of each function and write it as $y = f^{-1}(x)$.



38. **Splitting up curves** The equation $y^4 = 4\,x^2$ is associated with four one-to-one functions, $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ (see figure).

a. Find the domain and a formula for each function.

b. Find the inverse of each function and write it as $y = f^{-1}(x)$.



39–44. Derivatives of inverse functions at a point *Find the derivative of the inverse of the following functions at the specified point on the graph of the inverse function. You do not need to find* f^{-1}.

39. $f(x) = 3x + 4$; $(16, 4)$

40. $f(x) = \frac{1}{2}x + 8$; $(10, 4)$

41. $f(x) = -5x + 4$; $(-1, 1)$

42. $f(x) = x^2 + 1$, for $x \ge 0$; $(5, 2)$

43. $f(x) = \tan x$; $\left(1, \frac{\pi}{4}\right)$

44. $f(x) = x^2 - 2x - 3$, for $x \le 1$; $(12, -3)$

45–48. Slopes of tangent lines *Given the function f, find the slope of the line tangent to the graph of* f^{-1} *at the specified point on the graph of* f^{-1}.

45. $f(x) = \sqrt{x}$; $(2, 4)$

46. $f(x) = x^3$; $(8, 2)$

47. $f(x) = (x + 2)^2$, $x \ge -2$; $(36, 4)$

48. $f(x) = -x^2 + 8$, $x \ge 0$; $(7, 1)$

49–52. Derivatives and inverse functions

49. Find $\left(f^{-1}\right)'(3)$, where $f(x) = x^3 + x + 1$.

50. Suppose the slope of the curve $y = f^{-1}(x)$ at $(7, 4)$ is $\frac{2}{3}$. Find the slope of the curve $y = f^{-1}(x)$ at $(4, 7)$.

51. Suppose the slope of the curve $y = f^{-1}(x)$ at $(4, 7)$ is $\frac{4}{5}$. Find $f'(7)$.

52. Suppose the slope of the curve $y = f(x)$ at $(4, 7)$ is $\frac{1}{5}$. Find $\left(f^{-1}\right)'(7)$.

53–54. Derivatives of inverse functions from a table *Use the following tables to determine the indicated derivatives or state that the derivative cannot be determined.*

53.

x	-2	-1	0	1	2
$f(x)$	2	3	4	6	7
$f'(x)$	1	$\frac{1}{2}$	2	$\frac{3}{2}$	1

a. $\left(f^{-1}\right)'(4)$
b. $\left(f^{-1}\right)'(6)$
c. $\left(f^{-1}\right)'(1)$
d. $f'(1)$

54.

x	-4	-2	0	2	4
$f(x)$	0	1	2	3	4
$f'(x)$	5	4	3	2	1

a. $f'(f(0))$
b. $\left(f^{-1}\right)'(0)$
c. $\left(f^{-1}\right)'(1)$
d. $\left(f^{-1}\right)'(f(4))$

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $f(x) = x^2 + 1$, then $f^{-1}(x) = \frac{1}{x^2 + 1}$.

b. If $f(x) = \frac{1}{x}$, then $f^{-1}(x) = \frac{1}{x}$.

c. When restricted to the largest possible intervals, the function $f(x) = x^3 + x$ has three different inverses.

d. When restricted to the largest possible intervals, a tenth-degree polynomial could have at most ten different inverses.

e. If $f(x) = \frac{1}{x}$, then $\left(f^{-1}(x)\right)' = -\frac{1}{x^2}$.

T 56. Piecewise linear function Consider the function $f(x) = |x| - 2\,|x - 1|$.

a. Find the largest possible intervals on which f is one-to-one.

b. Find explicit formulas for the inverse of f on the intervals in part (a).

57–60. Finding all inverses *Find all the inverses associated with the following functions and state their domains.*

57. $f(x) = (x+1)^3$

58. $f(x) = (x-4)^2$

59. $f(x) = \dfrac{2}{x^2+2}$

60. $f(x) = \dfrac{2x}{x+2}$

61–68. Derivatives of inverse functions *Consider the following functions (on the given interval, if specified). Find the inverse function, express it as a function of x, and find the derivative of the inverse function.*

61. $f(x) = 3x - 4$

62. $f(x) = |x+2|$, for $x \le -2$

63. $f(x) = x^2 - 4$, for $x \ge 0$

64. $f(x) = \dfrac{x}{x+5}$

65. $f(x) = \sqrt{x+2}$

66. $f(x) = x^{2/3}$, for $x \ge 0$

67. $f(x) = x^{-1/2}$

68. $f(x) = x^3 + 3$

69–72. Geometry functions *Each function describes the volume V or surface area S of three-dimensional solids in terms of their radius. Find the inverse of each function that gives the radius in terms of V or S. Assume that r, V, and S are nonnegative. Express your answer in the form $r = f^{-1}(S)$ or $r = f^{-1}(V)$.*

69. Sphere: $V = \dfrac{4}{3}\pi r^3$

70. Sphere: $S = 4\pi r^2$

71. Cylinder with height 10: $V = 10\pi r^2$

72. Cone with height 12: $V = 4\pi r^2$

Explorations and Challenges »

T **73.** **Inverses of a quartic** Consider the quartic polynomial $y = f(x) = x^4 - x^2$.

a. Graph f and find the largest intervals on which it is one-to-one. The goal is to find the inverse function on each of these intervals.

b. Make the substitution $u = x^2$ to solve the equation $y = f(x)$ for x in terms of y. Be sure you have included all possible solutions.

c. Write the inverse function in the form $y = f^{-1}(x)$ for each of the intervals found in part (a).

74. Inverse of composite functions

a. Let $g(x) = 2x + 3$ and $h(x) = x^3$. Consider the composite function $f(x) = g(h(x))$. Find f^{-1} directly and then express it in terms of g^{-1} and h^{-1}.

b. Let $g(x) = x^2 + 1$ and $h(x) = \sqrt{x}$, for $x \geq 0$. Consider the composite function $f(x) = g(h(x))$. Find f^{-1} directly and then express it in terms of g^{-1} and h^{-1}.

c. Explain why if h and g are one-to-one, the inverse of $f(x) = g(h(x))$ exists.

75–76. Inverses of (some) cubics *Finding the inverse of a cubic polynomial is equivalent to solving a cubic equation. A special case that is simpler than the general case is the cubic $f(x) = x^3 + ax$. Find the inverse of the following cubics using the substitution (known as Vieta's substitution) $x = z - \frac{a}{3z}$. Be sure to determine where the function is one-to-one.*

75. $f(x) = x^3 + 2x$

76. $f(x) = x^3 + 4x - 1$

77. Tangents and inverses Suppose $y = L(x) = ax + b$ (with $a \neq 0$) is the equation of the line tangent to the graph of a one-to-one function f at (x_0, y_0). Also, suppose that $y = M(x) = cx + d$ is the equation of the line tangent to the graph of f^{-1} at (y_0, x_0).

a. Express a and b in terms of x_0 and y_0.

b. Express c in terms of a, and d in terms of a, x_0, and y_0.

c. Prove that $L^{-1}(x) = M(x)$.