6.7 Physical Applications

We conclude this chapter on applications of integration with several problems from physics and engineering. The physical themes in these problems are mass, work, force and pressure. The common mathematical theme is the use of the slice-and-sum strategy, which always leads to a definite integral.

Density and Mass »

Density is the concentration of mass in an object and is usually measured in units of mass per volume (for example, g/cm^3). An object with *uniform* density satisfies the basic relationship

$$\text{mass} = \text{density} \cdot \text{volume}.$$

When the density of an object *varies*, this formula no longer holds, and we must appeal to calculus.

In this section we introduce mass calculations for thin objects that can be viewed as line segments (such as wires or thin bars). The bar shown in **Figure 6.68** has a density ρ that varies along its length. For one-dimensional objects we use *linear density* with units of mass per length (for example, g/cm). What is the mass of such an object?

Figure 6.68

Note »

In Chapter 16, we return to mass calculations for two- and three-dimensional objects (plates and solids).

Quick Check 1 In Figure 6.68, suppose $a = 0$, $b = 3$, and the density of the rod in g/cm is $\rho(x) = (4 - x)$. (a) Where is the rod lightest and heaviest? (b) What is the density at the middle of the bar? ◆

Answer »

We begin by dividing the bar, represented by the interval $a \le x \le b$, into n subintervals of equal length $\Delta x = \dfrac{b - a}{n}$ (**Figure 6.69**).

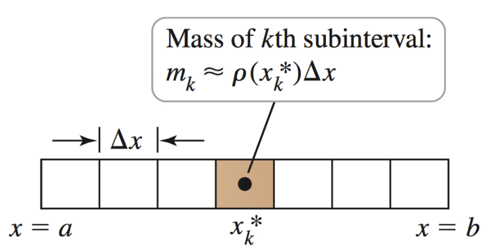


Figure 6.69

Let x_k^* be any point in the kth subinterval, for $k = 1, \ldots, n$. The mass of the kth segment of the bar, denoted m_k, is approximately the density at x_k^* multiplied by the length of the interval, or $m_k \approx \rho(x_k^*)\,\Delta x$. So the approximate mass of the entire bar is

$$\sum_{k=1}^{n} m_k \approx \sum_{k=1}^{n} \underbrace{\rho(x_k^*)\,\Delta x}_{m_k}.$$

The exact mass is obtained by taking the limit as $n \to \infty$ and as $\Delta x \to 0$, which produces a definite integral.

Note »

> Note that the units of the integral work out as they should: ρ has units of mass per length and dx has units of length; so $\rho(x)\,dx$ has units of mass.

DEFINITION Mass of a One-Dimensional Object

Suppose a thin bar or wire is represented by the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The **mass** of the object is

$$m = \int_a^b \rho(x)\,dx.$$

EXAMPLE 1 Mass from variable density

A thin, two-meter bar, represented by the interval $0 \le x \le 2$, is made of an alloy whose density in units of kg/m is given by $\rho(x) = \left(1 + x^2\right)$. What is the mass of the bar?

SOLUTION »

The mass of the bar in kilograms is

$$m = \int_a^b \rho(x)\,dx = \int_0^2 \left(1 + x^2\right) dx = \left(x + \frac{x^3}{3}\right)\Bigg|_0^2 = \frac{14}{3}.$$

Related Exercises 14–15 ◆

Quick Check 2 A thin bar occupies the interval $0 \le x \le 2$ and has a density in kg/m of $\rho(x) = \left(1 + x^2\right)$. Using the minimum value of the density, what is a lower bound for the mass of the object? Using the maximum value of the density, what is an upper bound for the mass of the object? ◆

Answer »

> Minimum mass = 2 kg; maximum mass = 10 kg

Work »

Work can be described as the change in energy when a force causes a displacement of an object. When you carry a refrigerator up a flight of stairs or push a stalled car, you apply a force that results in the displacement of an object, and work is done. If a *constant* force F displaces an object a distance d in the direction of the force, the work done is the force multiplied by the distance:

$$\text{work} = \text{force} \cdot \text{distance}.$$

It is easiest to use metric units for force and work. A newton (N) is the force required to give a 1-kg mass an acceleration of $1\ \text{m}/\text{s}^2$. A joule (J) is 1 newton-meter (N · m), the work done by a 1-N force acting over a distance of 1 m.

Calculus enters the picture with *variable* forces. Suppose an object is moved along the x-axis by a variable force F that is directed along the x-axis (**Figure 6.70**). How much work is done in moving the object between $x = a$ and $x = b$? Once again, we use the slice-and-sum strategy.

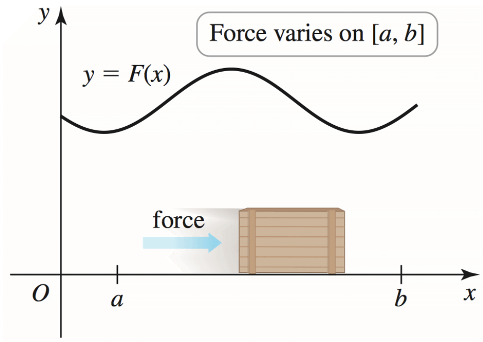


Figure 6.70

The interval $[a, b]$ is divided into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. We let x_k^* be any point in the kth subinterval, for $k = 1, \ldots, n$. On that subinterval the force is approximately constant with a value of $F(x_k^*)$. Therefore, the work done in moving the object across the kth subinterval is approximately $F(x_k^*)\,\Delta x$ (force $\cdot$ distance). Summing the work done over each of the n subintervals, the total work over the interval $[a, b]$ is approximately

$$W \approx \sum_{k=1}^{n} F(x_k^*)\,\Delta x.$$

This approximation becomes exact when we take the limit as $n \to \infty$ and $\Delta x \to 0$. The total work done is the integral of the force over the interval $[a, b]$ (or, equivalently, the net area under the force curve in Figure 6.70).

Quick Check 3 Explain why the sum of the work over n subintervals is only an approximation to the total work. ◆

Answer »

We assume that the force is constant over each subinterval, when, in fact, it varies over each subinterval.

DEFINITION **Work**

The work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

An application of force and work that is easy to visualize is the stretching and compression of a spring. Suppose an object is attached to a spring on a frictionless horizontal surface; the object slides back and forth under the influence of the spring. We say that the spring is at *equilibrium* when it is neither compressed nor stretched. It is convenient to let x be the position of the object, where $x = 0$ is the equilibrium position (**Figure 6.71**).

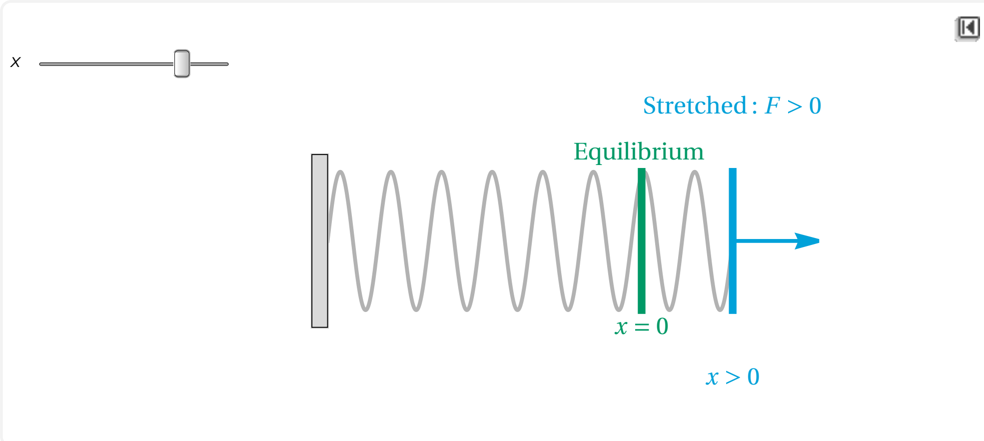


Figure 6.71

According to **Hooke's law**, the force required to keep the spring in a compressed or stretched position x units from the equilibrium position is $F(x) = k\,x$, where the spring constant k measures the stiffness of the spring. Note that to stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive direction) is required. To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required (**Figure 6.72**). In other words, the force required to displace the spring is always in the direction of the displacement.

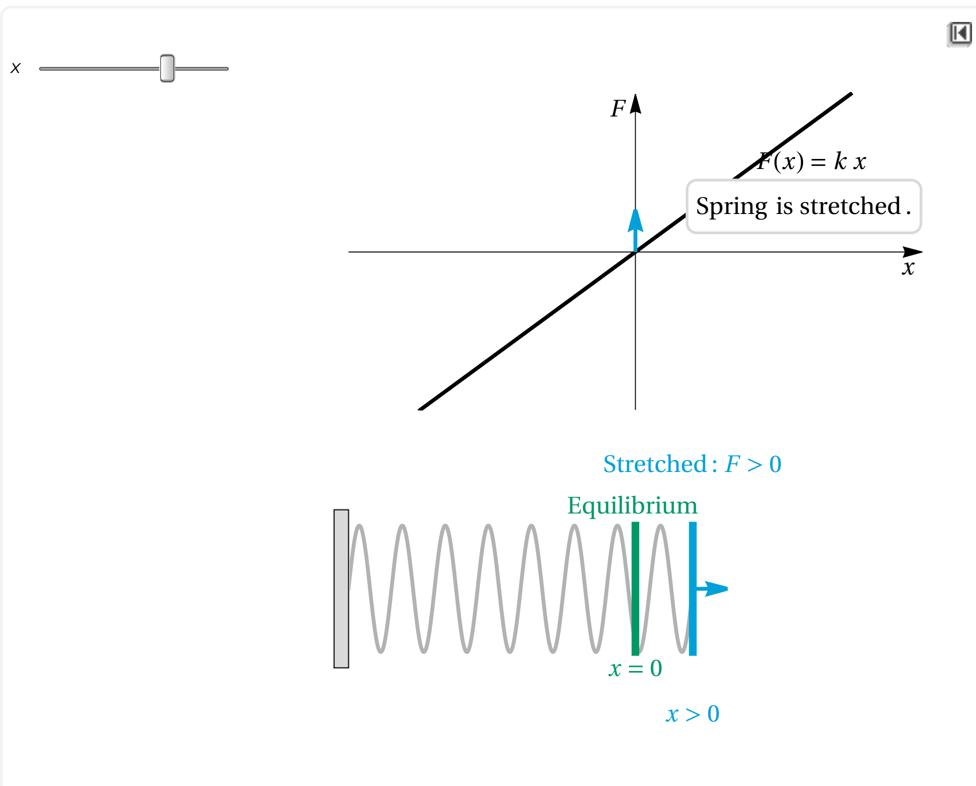


Figure 6.72

Note »

> Hooke's law was proposed by the English scientist Robert Hooke (1635-1703), who also coined the biological term *cell.* Hooke's law works well for springs made of many common materials. However, some springs obey more complicated nonlinear spring laws (see Exercise 59).

EXAMPLE 2 **Compressing a spring**

Suppose a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position and hold it in that position.

a. Assuming the spring obeys Hooke's law, find the spring constant k.

b. How much work is needed to *compress* the spring 0.5 m from its equilibrium position?

c. How much work is needed to *stretch* the spring 0.25 m from its equilibrium position?

d. How much additional work is required to stretch the spring 0.25 m if it has already been stretched 0.1 m from its equilibrium position?

SOLUTION »

a. The fact that a force of 10 N is required to keep the spring stretched at $x = 0.1$ m means (by Hooke's law) that $F(0.1) = k\,(0.1\text{ m}) = 10$ N. Solving for the spring constant, we find that $k = 100$ N/m. Therefore, Hooke's law for this spring is $F(x) = 100\,x$.

b. The work in joules required to compress the spring from $x = 0$ to $x = -0.5$ is

$$W = \int_a^b F(x)\,dx = \int_0^{-0.5} 100\,x\,dx = 50\,x^2\Big|_0^{-0.5} = 12.5.$$

Note »

c. The work in joules required to stretch the spring from $x = 0$ to $x = 0.25$ is

$$W = \int_a^b F(x)\,dx = \int_0^{0.25} 100\,x\,dx = 50\,x^2\Big|_0^{0.25} = 3.125.$$

d. The work in joules required to stretch the spring from $x = 0.1$ to $x = 0.35$ is

$$W = \int_a^b F(x)\,dx = \int_{0.1}^{0.35} 100\,x\,dx = 50\,x^2\Big|_{0.1}^{0.35} = 5.625.$$

Comparing parts (c) and (d), we see that more work is required to stretch the spring 0.25 m starting at $x = 0.1$ than starting at $x = 0$.

Related Exercises 23–24 ◆

Quick Check 4 In Example 2, explain why more work is needed in part (d) than in part (c), even though the displacement is the same. ◆

Answer »

> The restoring force of the spring increases as the spring is stretched $(F(x) = 100\,x)$. Greater restoring forces are encountered on the interval $[0.1, 0.35]$ than on the interval $[0, 0.25]$.

Lifting Problems

Another common work problem arises when the motion is vertical and the force is due to gravity. The gravitational force exerted on an object with mass m (measured in kg) is $F = m\,g$, where $g = 9.8\text{ m}/\text{s}^2$ is the acceleration due to gravity near the surface of Earth. The work in joules required to lift an object of mass m a vertical distance of y meters is

$$\text{work} = \text{force} \cdot \text{distance} = m\,g\,y.$$

This type of problem becomes interesting when the object being lifted is a rope, a chain, or a cable. In these situations, different parts of the object are lifted different distances, so integration is necessary.

Imagine a chain of length L meters with a constant density of ρ kg/m hanging vertically from a scaffolding platform at a construction site. To compute the work done in lifting the chain to the platform, we introduce a coordinate system where $y = 0$ corresponds to the bottom of the chain and $y = L$ corresponds to the top of the chain (**Figure 6.73**). We then divide the interval $[0, L]$ into n subintervals of equal length Δy and choose a point y_k^* from each subinterval $[y_{k-1}, y_k]$, for $k = 1, \ldots, n$.

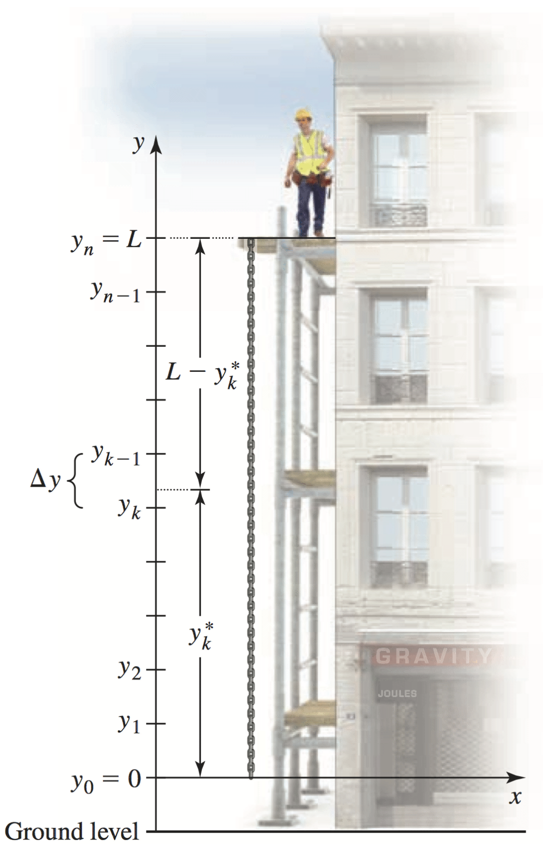


Figure 6.73

The mass of each segment of the chain is $m = \rho\,\Delta y$, and the point y_k^* in the kth segment is lifted a distance of $L - y_k^*$. Therefore, the work required to lift the kth segment is approximately

$$W_k = \underbrace{\rho\,\Delta y\,g}_{\text{force}} \cdot \underbrace{(L - y_k^*)}_{\text{distance}},$$

so the total work required to lift the chain is

$$W = \sum_{k=1}^{n} \rho\,g\,(L - y_k^*)\,\Delta y.$$

As the length of each segment Δy tends to zero and the number of segments tends to infinity, we obtain a definite integral for the total work in the limit:

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} \rho\, g\,(L - y_k^*)\,\Delta y = \int_0^L \rho\, g\,(L-y)\,dy. \qquad (1)$$

The function $L - y$ in equation (1) measures the distance through which a point y on the chain moves. Recognize that this function changes depending on the location of the origin relative to the chain, as illustrated in Example 3 (and Quick Check 6).

Note

> The units of ρ, Δy, g, and $L - y_k^*$ are kg/m, m, m/s^2, and m, respectively. When multiplied, they yield $\dfrac{\text{kg m}^2}{\text{s}^2}$, which are the units of a joule.

EXAMPLE 3 **Lifting a chain and bucket**

A ten-meter chain with density of 1.5 kg/m hangs from a platform at a construction site that is 11 meters above the ground (**Figure 6.74a**).

a. Compute the work required to lift the chain to the platform.

b. Several packages of nails are placed in a one-meter-tall bucket that rests on the ground; the mass of the bucket and nails together is 15 kg, and the chain is attached to the bucket (**Figure 6.74b**). How much work is required to lift the bucket to the platform?

Figure 6.74

SOLUTION

a. We use a coordinate system with $y = 0$ placed at the bottom of the chain. By equation (1), the work in joules required to lift the chain to the platform is

$$\begin{aligned} W &= \int_0^L \rho\, g\,(L - y)\, dy \\ &= \int_0^{10} 1.5\, g\,(10 - y)\, dy && L = 10 \text{ m};\ \rho = 1.5 \text{ kg/m} \\ &= 1.5\, g \left(10\, y - \frac{y^2}{2}\right)\Bigg|_0^{10} && \text{Integrate.} \\ &= 1.5\, g \left(100 - \frac{100}{2}\right) && \text{Evaluate.} \\ &= 735. && \text{Simplify; } g = 9.8 \text{ m}/s. \end{aligned}$$

Had we instead chosen a coordinate system with $y = 0$ corresponding to the ground, the work integral would be

$$W = \int_1^{11} 1.5\, g\, (11 - y)\, dy$$

because the chain would then lie in the interval [1, 11], and a point y on the chain would move a distance $(11 - y)$ m before reaching the platform. You can verify that the value of this definite integral is also 735 joules.

b. This time we use a coordinate system with $y = 0$ placed on the ground at the bottom of the bucket; we compute the work required to lift the chain and bucket separately. As shown in part (a), the work needed to lift the chain is 735 joules. Integration is not necessary to compute the work needed to lift the bucket (see Quick Check 5). We simply note that the bucket has a mass of 15 kg and every point in the bucket moves a distance of 11 m, so the work in joules required to lift it is

$$W = m\, g\, y = 15\, (9.8)\, (11) = 1617.$$

The work required to lift both the chain and the bucket is 735 + 1617 = 2352 joules.

Related Exercises 31–32 ◆

Quick Check 5 In Example 3b, the bucket occupies the interval [0, 1] and the chain occupies the interval [1, 11] (Figure 6.74b). Why is integration used to compute the work needed to lift the chain but not to compute the work needed to lift the bucket? ◆

Answer »

> The chain is a flexible object, and different points on the chain are lifted different distances as it is pulled to the platform. The bucket, however, is a rigid object, and every point in the bucket is lifted 11 m when it goes from resting on the ground to resting on the platform (in other words, it can be treated as a point mass).

Quick Check 6 Set up and evaluate the work integral in Example 3a using a coordinate system with $y = 0$ placed at the top of the chain and with a positive y-axis that points downward. ◆

Answer »

> $\int_0^{10} 1.5\, g\, y\, dy = 735$ joules.

Pumping Problems

The chains and cables encountered in the lifting problems from the previous pages can be modeled as one-dimensional objects. An extra layer of complexity is added when investigating the work required to pump fluid from a tank, although the same principles remain in play.

Suppose a fluid such as water is pumped out of a tank to a height h above the bottom of the tank. How much work is required, assuming the tank is full of water? Three key observations lead to the solution.

- Water from different levels of the tank is lifted different vertical distances, requiring different amounts of work.
- Water from the same horizontal plane is lifted the same distance, requiring the same amount of work.
- A volume V of water has mass ρV, where $\rho = 1\ \text{g}/\text{cm}^3 = 1000\ \text{kg}/\text{m}^3$ is the density of water.

To solve this problem, we let the y-axis point upward with $y = 0$ at the bottom of the tank. The body of water that must be lifted extends from $y = 0$ to $y = b$ (which *may* be the top of the tank). The level to which the water must be raised is $y = h$, where $h \geq b$ (**Figure 6.75**). We now slice the water into n horizontal layers, each having thickness Δy. The kth layer occupying the interval $[y_{k-1}, y_k]$, for $k = 1, \ldots, n$, is approximately y_k^* units above the bottom of the tank, where y_k^* is any point in $[y_{k-1}, y_k]$.

Note »

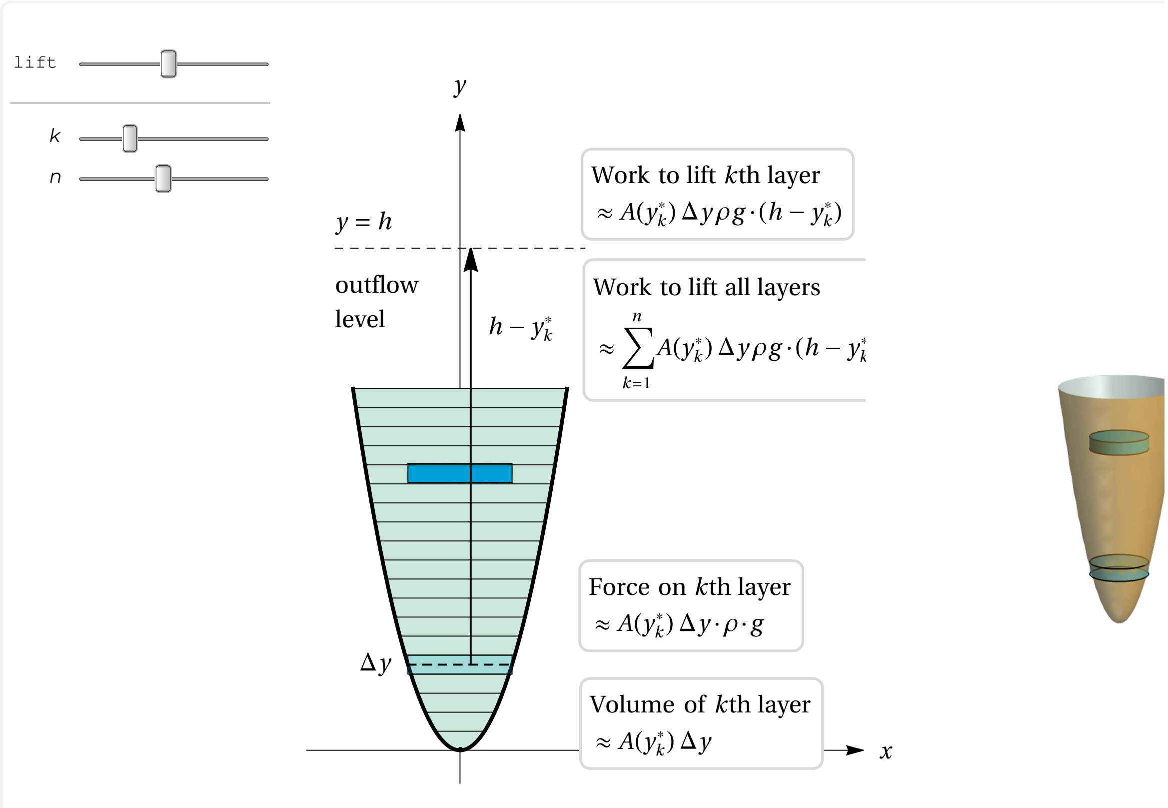


Figure 6.75

The cross-sectional area of the kth layer at y_k^*, denoted $A(y_k^*)$, is determined by the shape of the tank; the solution depends on being able to find A for all values of y. Because the volume of the kth layer is approximately $A(y_k^*)\,\Delta y$, the force on the kth layer (its weight) is

$$F_k = mg \approx \overbrace{\underbrace{A(y_k^*)\,\Delta y}_{\text{volume}} \cdot \underbrace{\rho}_{\text{density}}}^{\text{mass}} \cdot g.$$

To reach the level $y = h$, the kth layer is lifted an approximate distance of $(h - y_k^*)$ (Figure 6.75). So the work in lifting the kth layer to a height h is approximately

$$W_k = \underbrace{A(y_k^*)\,\Delta y\,\rho g}_{\text{force}} \cdot \underbrace{(h - y_k^*)}_{\text{distance}}.$$

Summing the work required to lift all the layers to a height h, the total work is

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} A(y_k^*)\,\rho\,g\,(h - y_k^*)\,\Delta y.$$

This approximation becomes more accurate as the width of the layers Δy tends to zero and the number of layers tends to infinity. In this limit, we obtain a definite integral from $y = 0$ to $y = b$. The total work required to empty the tank is

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} A(y_k^*)\, \rho\, g\, (h - y_k^*)\, \Delta y = \int_0^b \rho\, g\, A(y)\, (h-y)\, dy.$$

This derivation assumes the *bottom* of the tank is at $y = 0$, in which case the distance that the slice at level y must be lifted is $D(y) = h - y$. If you choose a different location for the origin, the function D will be different. Here is a general procedure for any choice of origin.

PROCEDURE **Solving Pumping Problems**

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.

2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is

$$W = \int_a^b \rho\, g\, A(y)\, D(y)\, dy.$$

Note »

Notice that the work integral for pumping problems reduces to the work integral for lifting problems given in equation (1) when we take $A(y) = 1$.

We now use this procedure to solve two pumping problems.

EXAMPLE 4 Pumping water

How much work is needed to pump all the water out of a cylindrical tank with a height of 10 m and a radius of 5 m? The water is pumped to an outflow pipe 15 m above the bottom of the tank.

SOLUTION »

Figure 6.76 shows the cylindrical tank filled to capacity and the outflow 15 m above the bottom of the tank.

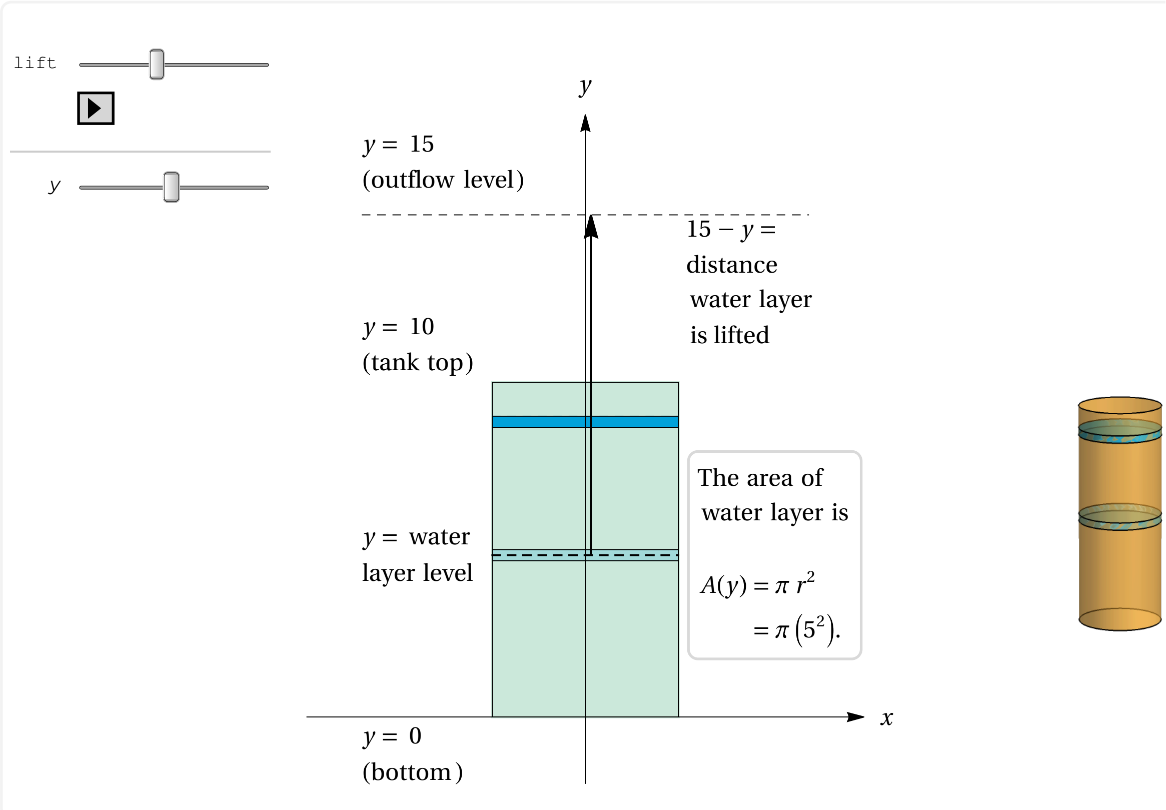


Figure 6.76

We let $y = 0$ represent the bottom of the tank and $y = 10$ represent the top of the tank. In this case, all horizontal slices are circular disks of radius $r = 5$ m. Therefore, for $0 \le y \le 10$, the cross-sectional area is

$$A(y) = \pi r^2 = \pi 5^2 = 25\pi.$$

Note that the water is pumped to a level $h = 15$ m above the bottom of the tank, so the lifting distance is $D(y) = 15 - y$. The resulting work integral is

$$W = \int_0^{10} \rho g \underbrace{A(y)}_{25\pi} \underbrace{D(y)}_{15-y} dy = 25\pi\rho g \int_0^{10} (15 - y)\, dy.$$

Substituting $\rho = 1000 \text{ kg}/\text{m}^3$ and $g = 9.8 \text{ m}/\text{s}^2$, the total work in joules is

$$\begin{aligned} W &= 25\pi\rho g \int_0^{10} (15 - y)\, dy \\ &= 25\pi \underbrace{(1000)}_{\rho} \underbrace{(9.8)}_{g} \left(15y - \frac{1}{2}y^2\right)\Bigg|_0^{10} \\ &\approx 7.7 \times 10^7. \end{aligned}$$

The work required to pump the water out of the tank is approximately 77 million joules.

Note »

Recall that $g \approx 9.8\ \mathrm{m/s^2}$. You should verify that the units are consistent in this calculation: The units of ρ, g, $A(y)$, $D(y)$, and dy are $\mathrm{kg/m^3}$, $\mathrm{m/s^2}$, $\mathrm{m^2}$, m, and m, respectively. The resulting units of W are $\mathrm{kg \cdot m^2/s^2}$, or J. A more convenient unit for large amounts of work and energy is the kilowatt-hr, which is 3.6 million joules.

Related Exercises 36–37 ◆

Quick Check 7 In the Example 4, how would the integral change if the outflow pipe were at the top of the tank? ◆

Answer »

The factor $(15 - y)$ in the integral is replaced by $(10 - y)$.

EXAMPLE 5 Pumping gasoline

A cylindrical tank with a length of 10 m and a radius of 5 m is on its side and half-full of gasoline (**Figure 6.77**). How much work is required to empty the tank through an outlet pipe at the top of the tank? (The density of gasoline is $\rho = 737\ \mathrm{kg/m^3}$.)

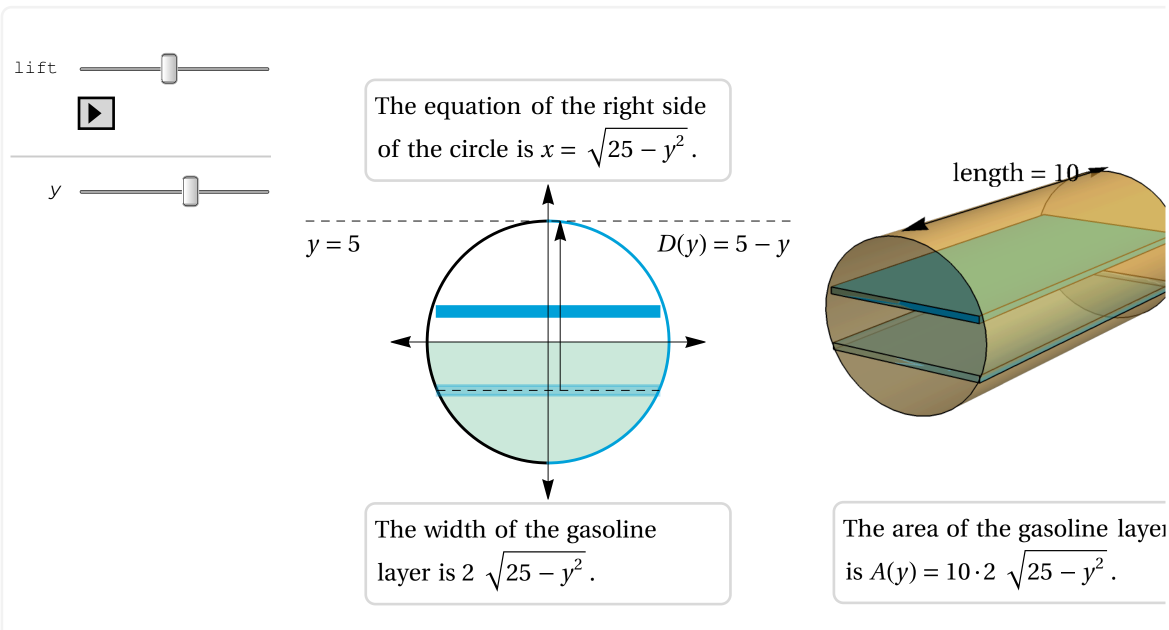


Figure 6.77

In this problem we choose a different origin by letting $y = 0$ and $y = -5$ correspond to the center and the bottom of the tank, respectively. For $-5 \le y \le 0$, a horizontal layer of gasoline located at a depth y is a rectangle with a length of 10 and width of $2\sqrt{25 - y^2}$ (Figure 6.77). Therefore, the cross-sectional area of the layer at depth y is

$$A(y) = 20\sqrt{25 - y^2}.$$

Note »

Again, there are several choices for the location of the origin. The location in this example makes $A(y)$ easy to compute.

The distance the layer at level y must be lifted to reach the top of the tank is $D(y) = 5 - y$, where $-5 \le y \le 0$; note that $5 \le D(y) \le 10$. The resulting work integral is

$$W = \underbrace{737}_{\rho}\,\underbrace{(9.8)}_{g}\int_{-5}^{0}\underbrace{20\sqrt{25-y^2}}_{A(y)}\,\underbrace{(5-y)}_{D(y)}\,dy = 144{,}452\int_{-5}^{0}\sqrt{25-y^2}\,(5-y)\,dy.$$

This integral is evaluated by splitting it into two pieces and recognizing that one piece is the area of a quarter circle of radius 5:

$$\int_{-5}^{0}\sqrt{25-y^2}\,(5-y)\,dy = 5\underbrace{\int_{-5}^{0}\sqrt{25-y^2}\,dy}_{\text{area of quarter circle}} - \underbrace{\int_{-5}^{0}y\sqrt{25-y^2}\,dy}_{\text{let } u=25-y^2;\ du=-2y\,dy}$$

$$= 5\cdot\frac{25\pi}{4} + \frac{1}{2}\int_{0}^{25}\sqrt{u}\,du$$

$$= \frac{125\pi}{4} + \frac{1}{3}u^{3/2}\Big|_{0}^{25} = \frac{375\pi + 500}{12}.$$

Multiplying this result by $20\rho g = 144{,}452$, we find that the work required is approximately 20.2 million joules.

Related Exercises 42–44 ◆

Force and Pressure »

Another application of integration deals with the force exerted on a surface by a body of water. Again, we need a few physical principles.

Pressure is a force per unit area, measured in units such as newtons per square meter ($\mathrm{N/m^2}$). For example, the pressure of the atmosphere on the surface of Earth is about $14\ \mathrm{lb/in^2}$ (approximately 100 kilopascals, or $10^5\ \mathrm{N/m^2}$). As another example, if you stood on the bottom of a swimming pool, you would feel pressure due to the weight (force) of the column of water above your head. If your head is flat and has surface area $A\ \mathrm{m^2}$ and it is h meters below the surface, then the column of water above your head has volume $Ah\ \mathrm{m^3}$. That column of water exerts a force:

$$F = \text{mass}\cdot\text{acceleration} = \underbrace{\text{volume}\cdot\text{density}}_{\text{mass}}\cdot g = Ah\rho g,$$

where ρ is the density of water and g is the acceleration due to gravity. Therefore, the pressure on your head is the force divided by the surface area of your head:

$$\text{pressure} = \frac{\text{force}}{A} = \frac{Ah\rho g}{A} = \rho g h.$$

This pressure is called **hydrostatic pressure** (meaning the pressure of *water at rest*), and it has the following important property: *It has the same magnitude in all directions*. Specifically, the hydrostatic pressure on a vertical wall of the swimming pool at a depth h is also $\rho g h$. This is the only fact needed to find the total force on vertical walls such as dams and swimming pools. We assume the water completely covers the face of the dam.

The first step in finding the force on the face of the dam is to introduce a coordinate system. We choose a y-axis pointing upward with $y = 0$ corresponding to the base of the dam and $y = a$ corresponding to the top of the dam (**Figure 6.78**). Because the pressure varies with depth (y-direction), the dam is sliced horizontally into

n strips of equal thickness Δy. The kth strip corresponds to the interval $[y_{k-1}, y_k]$, and we let y_k^* be any point in that interval. The depth of that strip is approximately $h = a - y_k^*$, so the hydrostatic pressure on that strip is approximately $\rho g (a - y_k^*)$.

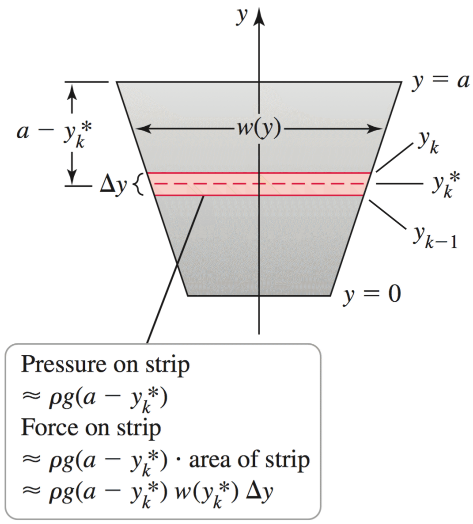


Figure 6.78

Note »

> We have chosen $y = 0$ to be the base of the dam. Depending on the geometry of the problem, it may be more convenient (less computation) to let $y = 0$ be at the top of the dam. Note that the depth function $D(y) = a - y$ varies with the chosen coordinate system. Experiment with different choices.

The crux of any dam problem is finding the width of the strips as a function of y, which we denote $w(y)$. Each dam has its own width function; however, once the width function is known, the solution follows directly. The approximate area of the kth strip is its width multiplied by its thickness, or $w(y_k^*)\,\Delta y$. The force on the kth strip (which is the area of the strip multiplied by the pressure) is approximately

$$F_k = \underbrace{\rho g (a - y_k^*)}_{\text{pressure}} \; \underbrace{w(y_k^*)\,\Delta y}_{\text{area of strip}}.$$

Summing the forces over the n strips, the total force is

$$F \approx \sum_{k=1}^{n} F_k = \sum_{k=1}^{n} \rho g (a - y_k^*)\, w(y_k^*)\,\Delta y.$$

To find the exact force, we let the thickness of the strips tend to zero and the number of strips tend to infinity, which produces a definite integral. The limits of integration correspond to the base $(y = 0)$ and top $(y = a)$ of the dam. Therefore, the total force on the dam is

$$F = \lim_{n\to\infty} \sum_{k=1}^{n} \rho g (a - y_k^*)\, w(y_k^*)\,\Delta y = \int_0^a \rho g (a - y)\, w(y)\, dy.$$

PROCEDURE **Solving Force-on-Dam Problems**

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).

2. Find the width function $w(y)$ for each value of y on the face of the dam.

3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho\, g\, \underbrace{(a - y)}_{\text{depth}}\, \underbrace{w(y)}_{\text{width}}\, dy.$$

EXAMPLE 6 Force on a dam

A large vertical dam in the shape of a symmetric trapezoid has a height of 30 m, a width of 20 m at its base, and a width of 40 m at the top (**Figure 6.79**). What is the total force on the face of the dam when the reservoir is full?

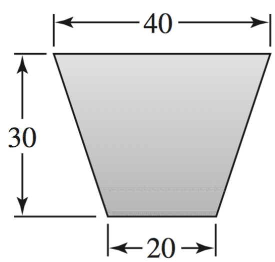


Figure 6.79

SOLUTION »

We place the origin at the center of the base of the dam (**Figure 6.80**).

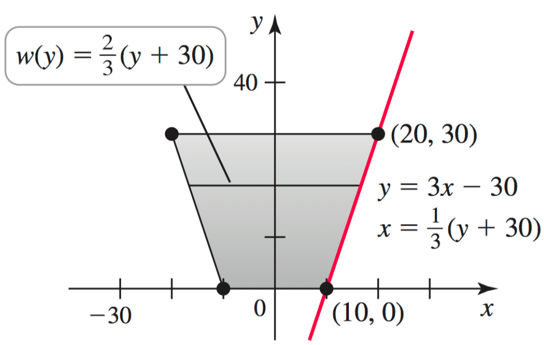


Figure 6.80

The right slanted edge of the dam is a segment of the line that passes through the points (10, 0) and (20, 30). An equation of that line is

$$y - 0 = \frac{30}{10}(x - 10) \quad \text{or} \quad y = 3x - 30 \quad \text{or} \quad x = \frac{1}{3}(y + 30).$$

Notice that at a depth of y, where $0 \le y \le 30$, the width of the dam is

$$w(y) = 2\,x = \frac{2}{3}(y + 30).$$

Note »

Using $\rho = 1000\ \text{kg}/\text{m}^3$ and $g = 9.8\ \text{m}/\text{s}^2$, the total force on the dam (in newtons) is

$$\begin{aligned}
F &= \int_0^a \rho\, g\,(a - y)\, w(y)\, dy && \text{Force integral}\\
&= \rho\, g \int_0^{30} \underbrace{(30 - y)}_{a-y}\, \underbrace{\frac{2}{3}(y + 30)}_{w(y)}\, dy && \text{Substitute.}\\
&= \frac{2}{3}\rho\, g \int_0^{30} \left(900 - y^2\right) dy && \text{Simplify.}\\
&= \frac{2}{3}\rho\, g \left(900\, y - \frac{y^3}{3}\right)\Bigg|_0^{30} && \text{Fundamental Theorem}\\
&\approx 1.18 \times 10^8.
\end{aligned}$$

The force of 1.18×10^8 N on the dam amounts to about 26 million pounds, or 13,000 tons.

Related Exercises 46–47 ◆

Exercises »

Getting Started »

Practice Exercises »

13–20. Mass of one–dimensional objects *Find the mass of the following thin bars with the given density function.*

13. $\rho(x) = 1 + \sin x$, for $0 \le x \le \pi$

14. $\rho(x) = 1 + x^3$, for $0 \le x \le 1$

15. $\rho(x) = 2 - \frac{x}{2}$, for $0 \le x \le 2$

16. $\rho(x) = 1 + 3\sin x$, for $0 \le x \le \pi$

17. $\rho(x) = x\sqrt{2 - x^2}$, for $0 \le x \le 1$

18. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 2 & \text{if } 2 < x \le 3 \end{cases}$

19. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } 2 < x \le 4 \end{cases}$

20. $\rho(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x\,(2 - x) & \text{if } 1 < x \le 2 \end{cases}$

21. Work from force How much work is required to move an object from $x = 0$ to $x = 3$ (measured in meters) in the presence of a force (in N) given by $F(x) = 2\,x$ acting along the x-axis?

22. Work from force How much work is required to move an object from $x = 1$ to $x = 3$ (measured in meters) in the presence of a force (in N) given by $F(x) = \dfrac{2}{x^2}$ acting along the x-axis?

23. Compressing and stretching a spring Suppose a force of 30 N is required to stretch and hold a spring 0.2 m from its equilibrium position.

- **a.** Assuming the spring obeys Hooke's law, find the spring constant k.
- **b.** How much work is required to compress the spring 0.4 m from its equilibrium position?
- **c.** How much work is required to stretch the spring 0.3 m from its equilibrium position?
- **d.** How much additional work is required to stretch the spring 0.2 m if it has already been stretched 0.2 m from its equilibrium position?

24. Compressing and stretching a spring Suppose a force of 15 N is required to stretch and hold a spring 0.25 m from its equilibrium position.

- **a.** Assuming the spring obeys Hooke's law, find the spring constant k.
- **b.** How much work is required to compress the spring 0.2 m from its equilibrium position?
- **c.** How much additional work is required to stretch the spring 0.3 m if it has already been stretched 0.25 m from its equilibrium position?

25. Work done by a spring A spring on a horizontal surface can be stretched and held 0.5 m from its equilibrium position with a force of 50 N.

- **a.** How much work is done in stretching the spring 1.5 m from its equilibrium position?
- **b.** How much work is done in compressing the spring 0.5 m from its equilibrium position?

26. Shock absorber A heavy-duty shock absorber is compressed 2 cm from its equilibrium position by a mass of 500 kg. How much work is required to compress the shock absorber 4 cm from its equilibrium position? (A mass of 500 kg exerts a force (in newtons) of 500 g, where $g = 9.8\ \mathrm{m}/\mathrm{s}^2$.)

27. Calculating work for different springs Calculate the work required to stretch the following springs 0.5 m from their equilibrium positions. Assume Hooke's law is obeyed.

- **a.** A spring that requires a force of 50 N to be stretched 0.2 m from its equilibrium position.
- **b.** A spring that requires 50 J of work to be stretched 0.2 m from its equilibrium position.

28. Calculating work for different springs Calculate the work required to stretch the following springs 0.4 m from their equilibrium positions. Assume Hooke's law is obeyed.

- **a.** A spring that requires a force of 50 N to be stretched 0.1 m from its equilibrium position.
- **b.** A spring that requires 2 J of work to be stretched 0.1 m from its equilibrium position.

29. Calculating work for different springs Calculate the work required to stretch the following springs 1.25 m from their equilibrium positions. Assume Hooke's law is obeyed.

- **a.** A spring that requires 100 J of work to be stretched 0.5 m from its equilibrium position
- **b.** A spring that requires a force of 250 N to be stretched 0.5 m from its equilibrium position

30. Work function A spring has a restoring force given by $F(x) = 25\,x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x = 0)$ to a variable distance x. Find and graph the work function. Compare the work required to stretch the spring x units from equilibrium to the work required to compress the spring x units from equilibrium.

31. Winding a chain A 30-m-long chain hangs vertically from a cylinder attached to a winch. Assume there is no friction in the system and the chain has a density of 5 kg/m.

a. How much work is required to wind the entire chain onto the cylinder using the winch?

b. How much work is required to wind the chain onto the cylinder if a 50-kg block is attached to the end of the chain?

32. Coiling a rope A 60-m-long, 9.4-mm-diameter rope hangs freely from a ledge. The density of the rope is 55 g/m. How much work is needed to pull the entire rope to the ledge?

33. Winding part of a chain A 20-m-long, 50-kg chain hangs vertically from a cylinder attached to a winch. How much work is needed to wind half of the chain onto the winch?

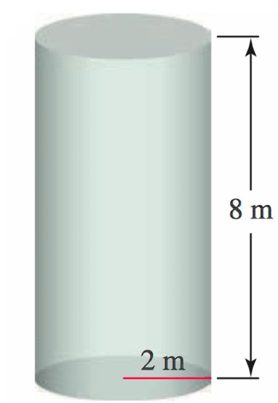
34. Leaky bucket A 1-kg bucket resting on the ground contains 3 kg of water. How much work is required to raise the bucket vertically a distance of 10 m if water leaks out of the bucket at a constant rate of $\frac{1}{5}$ kg/m? Assume the weight of the rope used to raise the bucket is negligible. (*Hint*: Use the definition of work, $W = \int_a^b F(y)\,dy$, where F is the variable force required to lift an object along a vertical line from $y = a$ to $y = b$.)

35. Emptying a swimming pool A swimming pool has the shape of a box with a base that measures 25 m by 15 m and a uniform depth of 2.5 m. How much work is required to pump the water out of the pool (to the level of the top of the pool) when it is full?

36. Emptying a cylindrical tank A cylindrical water tank has height 8 m and radius 2 m (see figure).

a. If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b. Is it true that it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.



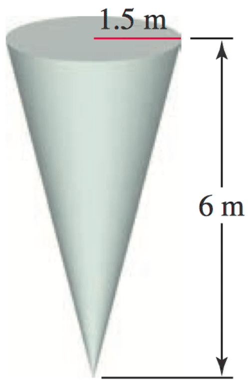
37. Emptying a half-full cylindrical tank Suppose the water tank in Exercise 36 is half-full of water. Determine the work required to empty the tank by pumping the water to a level 2 m above the top of the tank.

38. Emptying a partially filled swimming pool If the water in the swimming pool in Exercise 35 is 2 m deep, then how much work is required to pump all the water to a level 3 m above the bottom of the pool?

39. Emptying a conical tank A water tank is shaped like an inverted cone with height 6 m and base radius 1.5 m (see figure).

a. If the tank is full, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b. Is it true that it takes half as much work to pump the water out of the tank when it is filled to half its depth as when it is full? Explain.



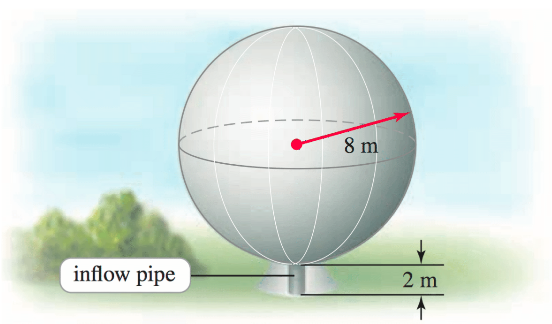
40. Upper and lower half A cylinder with height 8 m and radius 3 m is filled with water and must be emptied through an outlet pipe 2 m above the top of the cylinder.

a. Compute the work required to empty the water in the top half of the tank.

b. Compute the work required to empty the (equal amount of) water in the lower half of the tank.

c. Interpret the results of parts (a) and (b).

41. Filling a spherical tank A spherical water tank with an inner radius of 8 m has its lowest point 2 m above the ground. It is filled by a pipe that feeds the tank at its lowest point (see figure). Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

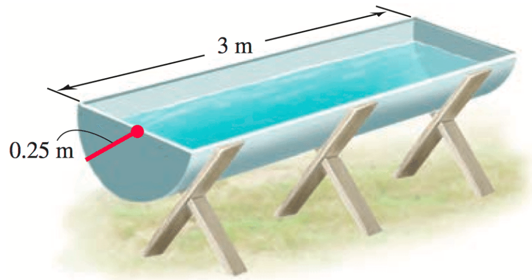


42. Emptying a water trough A water trough has a semicircular cross section with a radius of 0.25 m and a length of 3 m (see figure).

a. How much work is required to pump the water out of the trough (to the level of the top of the trough) when it is full?

b. If the length is doubled, is the required work doubled? Explain.

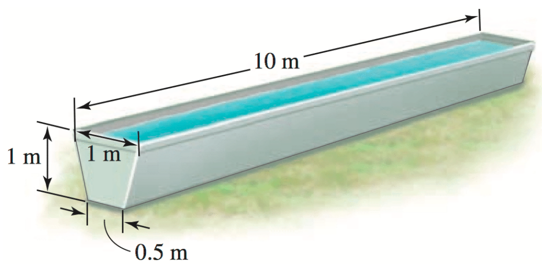
c. If the radius is doubled, is the required work doubled? Explain.



43. Emptying a water trough A cattle trough has a trapezoidal cross section with a height of 1 m and horizontal sides of length $\frac{1}{2}$ m and 1 m. Assume the length of the trough is 10 m (see figure).

a. How much work is required to pump the water out of the trough (to the level of the top of the trough) when it is full?

b. If the length is doubled, is the required work doubled? Explain.

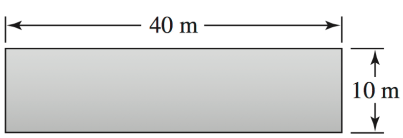


44. Pumping water Suppose the tank in Example 5 is full of water (rather than half full of gasoline). Determine the work required to pump all the water to an outlet pipe 15 m above the bottom of the tank.

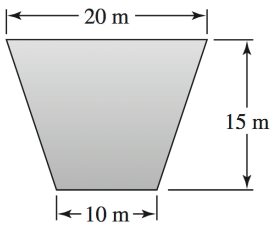
45. Emptying a conical tank An inverted cone (base above the vertex) is 2 m high and has a base radius of $\frac{1}{2}$ m. If the tank is full, how much work is required to pump the water to a level 1 m above the top of the tank?

46–49. Force on dams *The following figures show the shape and dimensions of small dams. Assuming the water level is at the top of the dam, find the total force on the face of the dam.*

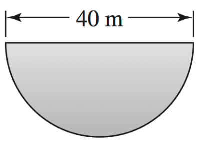
46.



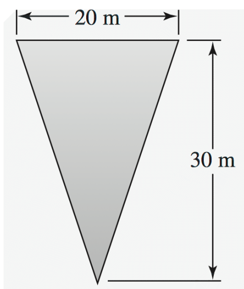
47.



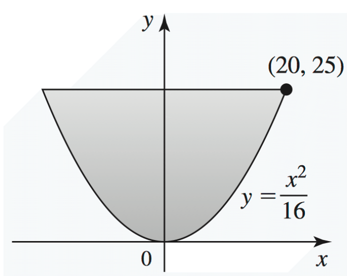
48.



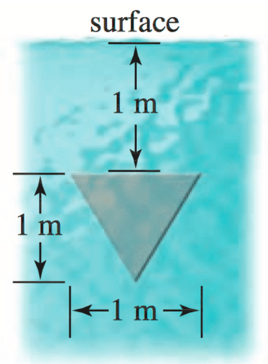
49.



50. Parabolic dam The lower edge of a dam is defined by the parabola $y = \frac{x^2}{16}$ (see figure). Use a coordinate system with $y = 0$ at the bottom of the dam to determine the total force of the water on the dam. Lengths are measured in meters. Assume the water level is at the top of the dam.



51. Force on a triangular plate A plate shaped like an isosceles triangle with a height of 1 m is placed on a vertical wall 1 m below the surface of a pool filled with water (see figure). Compute the force on the plate.



52–54. Force on a window *A diving pool that is* 4 m *deep and full of water has a viewing window on one of its vertical walls. Find the force on the following windows.*

52. The window is a square, 0.5 m on a side, with the lower edge of the window on the bottom of the pool.

53. The window is a square, 0.5 m on a side, with the lower edge of the window 1 m from the bottom of the pool.

54. The window is circular, with a radius of 0.5 m, tangent to the bottom of the pool.

55. **Force on a building** A large building shaped like a box is 50 m high with a face that is 80 m wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where y is the height above the ground. Calculate the total force on the building, which is a measure of the resistance that must be included in the design of the building.

56. **Force on the end of a tank** Determine the force on a circular end of the tank in Figure 6.77 if the tank is full of gasoline. The density of gasoline is $\rho = 737\ kg/m^3$.

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The mass of a thin wire is the length of the wire times its average density over its length.

b. The work required to stretch a linear spring (that obeys Hooke's law) 100 cm from equilibrium is the same as the work required to compress it 100 cm from equilibrium.

c. The work required to lift a 10-kg object vertically 10 m is the same as the work required to lift a 20-kg object vertically 5 m.

d. The total force on a 10-ft^2 region on the (horizontal) floor of a pool is the same as the total force on a 10-ft^2 region on a (vertical) wall of the pool.

Explorations and Challenges »

58. **Mass of two bars** Two bars of length L have densities $\rho_1(x) = 4(x+1)^{-2}$ and $\rho_2(x) = 6(x+1)^{-3}$, for $0 \le x \le L$.

a. For what values of L is bar 1 heavier than bar 2?

b. As the lengths of the bars increase, do their masses increase without bound? Explain.

59. A nonlinear spring Hooke's law is applicable to idealized (linear) springs that are not stretched or compressed too far from their equilibrium positions. Consider a nonlinear spring whose restoring force is given by $F(x) = 16x - 0.1x^3$, for $|x| \le 7$.

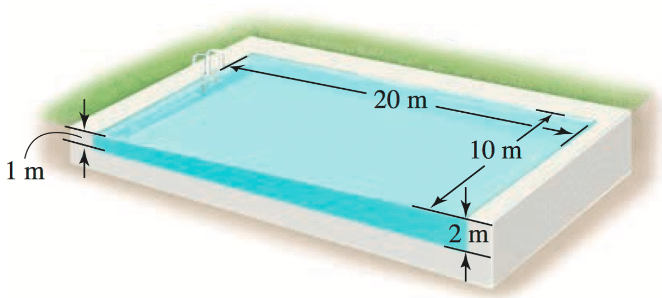
- **a.** Graph the restoring force and interpret it.
- **b.** How much work is done in stretching the spring from its equilibrium position $(x = 0)$ to $x = 1.5$?
- **c.** How much work is done in compressing the spring from its equilibrium position $(x = 0)$ to $x = -2$?

60. A vertical spring A 10-kg mass is attached to a spring that hangs vertically and is stretched 2 m from the equilibrium position of the spring. Assume a linear spring with $F(x) = kx$.

- **a.** How much work is required to compress the spring and lift the mass 0.5 m?
- **b.** How much work is required to stretch the spring and lower the mass 0.5 m?

61. Leaky cement bucket A 350 kg-bucket containing 4650 kg of cement is resting on the ground when a crane begins lifting it at a constant rate of 5 m/min. As the crane raises the bucket, cement leaks out of the bucket at a constant rate of 100 kg/min. How much work is required to lift the bucket a distance of 30 m if we ignore the weight of the crane cable attached to the bucket?

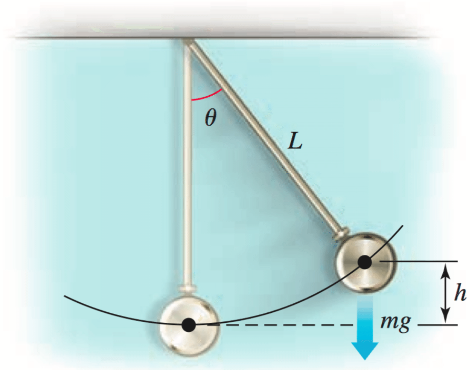
62. Emptying a real swimming pool A swimming pool is 20 m long and 10 m wide, with a bottom that slopes uniformly from a depth of 1 m at one end to a depth of 2 m at the other end (see figure). Assuming the pool is full, how much work is required to pump the water to a level 0.2 m above the top of the pool?



63. Drinking juice A glass has circular cross sections that taper (linearly) from a radius of 5 cm at the top of the glass to a radius of 4 cm at the bottom. The glass is 15 cm high and full of orange juice. How much work is required to drink all the juice through a straw if your mouth is 5 cm above the top of the glass? Assume the density of orange juice equals the density of water.

64. Lifting a pendulum A body of mass m is suspended by a rod of length L that pivots without friction (see figure). The mass is slowly lifted along a circular arc to a height h.

- **a.** Assuming the only force acting on the mass is the gravitational force, show that the component of this force acting along the arc of motion is $F = mg\sin\theta$.
- **b.** Noting that an element of length along the path of the pendulum is $ds = L\,d\theta$, evaluate an integral in θ to show that the work done in lifting the mass to a height h is mgh.

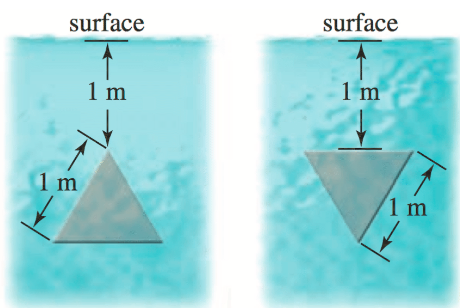


65. Critical depth A large tank has a plastic window on one wall that is designed to withstand a force of 90,000 N. The square window is 2 m on a side, and its lower edge is 1 m from the bottom of the tank.

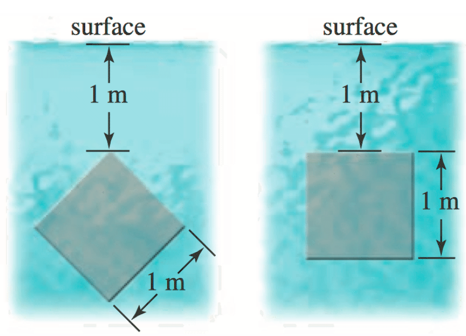
a. If the tank is filled to a depth of 4 m, will the window withstand the resulting force?

b. What is the maximum depth to which the tank can be filled without the window failing?

66. Orientation and force A plate shaped like an equilateral triangle 1 m on a side is placed on a vertical wall 1 m below the surface of a pool filled with water. On which plate in the figure is the force greater? Try to anticipate the answer and then compute the force on each plate.



67. Orientation and force A square plate 1 m on a side is placed on a vertical wall 1 m below the surface of a pool filled with water. On which plate in the figure is the force greater? Try to anticipate the answer and then compute the force on each plate.



68. Work by two different integrals A rigid body with a mass of 2 kg moves along a line due to a force that produces a position function $x(t) = 4t^2$, where x is measured in meters and t is measured in seconds. Find the work done during the first 5 s in two ways.

a. Note that $x''(t) = 8$; then use Newton's second law ($F = m\,a = m\,x''(t)$) to evaluate the work integral $W = \int_{x_0}^{x_f} F(x)\,dx$, where x_0 and x_f are the initial and final positions, respectively.

b. Change variables in the work integral and integrate with respect to t. Be sure your answer agrees with part (a).

69. Work in a gravitational field For large distances from the surface of Earth, the gravitational force is given by $F(x) = \dfrac{G\,M\,m}{(x+R)^2}$, where $G = 6.7 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2$ is the gravitational constant, $M = 6 \times 10^{24}$ kg is the mass of Earth, m is the mass of the object in the gravitational field, $R = 6.378 \times 10^6$ m is the radius of Earth, and $x \geq 0$ is the distance above the surface of Earth (in meters).

a. How much work is required to launch a rocket with a mass of 500 kg in a vertical flight path to a height of 2500 km (from Earth's surface)?

b. Find the work required to launch the rocket to a height of x kilometers, for $x > 0$.

c. How much work is required to reach outer space ($x \to \infty$)?

d. Equate the work in part (c) to the initial kinetic energy of the rocket, $\dfrac{1}{2}\,m\,v^2$, to compute the escape velocity of the rocket.

70. Buoyancy Archimedes' principle says that the buoyant force exerted on an object that is (partially or totally) submerged in water is equal to the weight of the water displaced by the object (see figure). Let $\rho_w = 1\ \text{g}/\text{cm}^3 = 1000\ \text{kg}/\text{m}^3$ be the density of water and let ρ be the density of an object in water. Let $f = \dfrac{\rho}{\rho_w}$. If $0 < f \leq 1$, then the object floats with a fraction f of its volume submerged; if $f > 1$, then the object sinks. Consider a cubical box with sides 2 m long floating in water with one-half of its volume submerged ($\rho = \dfrac{\rho_w}{2}$). Find the force required to fully submerge the box (so its top surface is at the water level).

(See the Guided Project *Buoyancy and Archimedes' Principle* for further explorations of buoyancy problems.)