6.6 Surface Area

In Sections 6.3 and 6.4, we introduced solids of revolution and presented methods for computing the volume of such solids. We now consider a related problem: computing the *area* of the surface of a solid of revolution. Surface area calculations are important in aerodynamics (computing the lift on an airplane wing) and biology (computing transport rates across cell membranes), to name just two applications. Here is an interesting observation: A surface area problem is “between” a volume problem (which is three-dimensional) and an arc length problem (which is one-dimensional). For this reason, you will see ideas that appear in both volume and arc length calculations as we develop the surface area integral.

Some Preliminary Calculations »

Consider a curve $y = f(x)$ on an interval $[a, b]$, where f is both differentiable and positive on $[a, b]$. Now imagine revolving the curve about the x-axis to generate a *surface of revolution* (**Figure 6.59**). Our objective is to find the area of this surface.

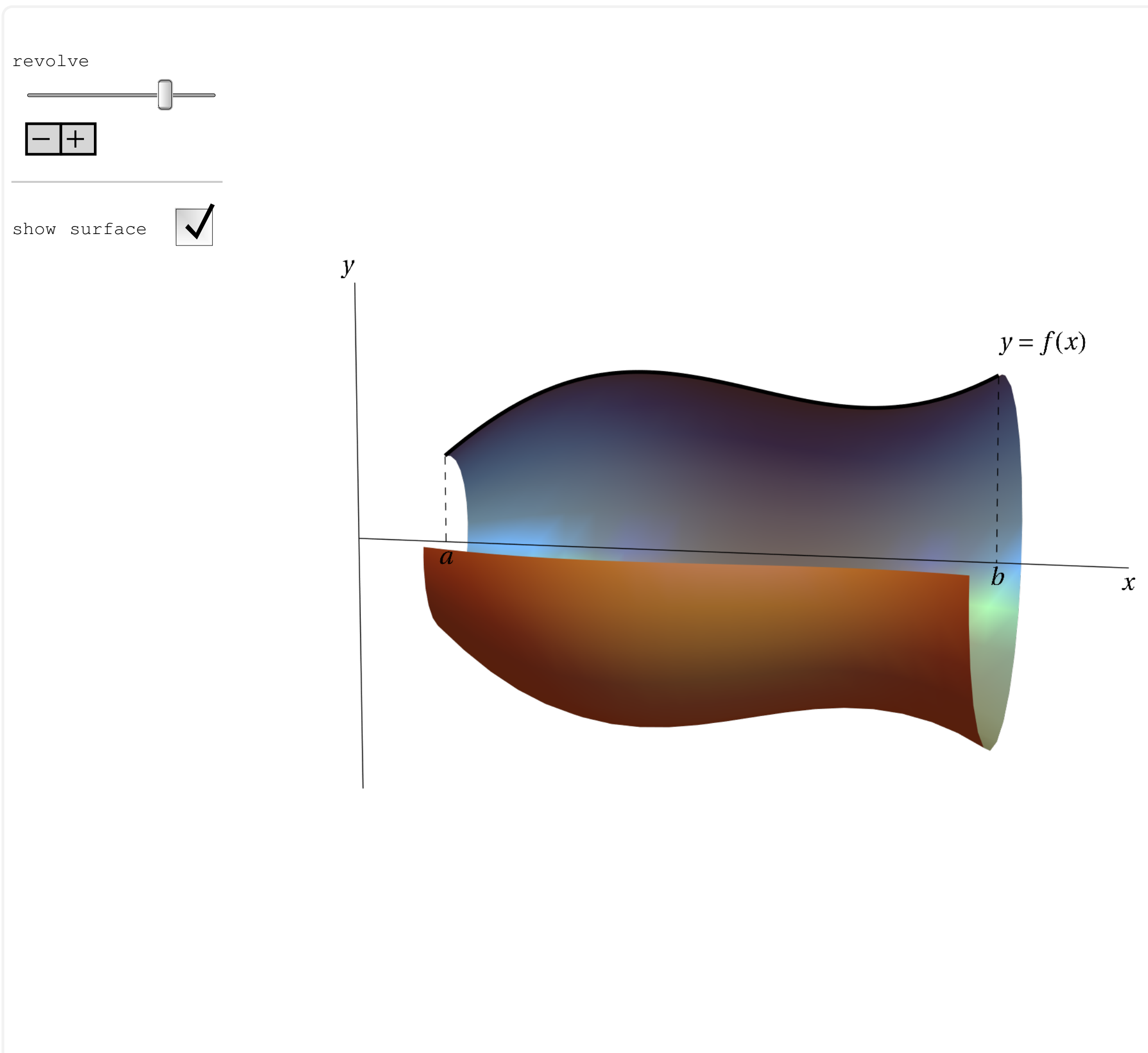


Figure 6.59

Before tackling this problem, we consider a preliminary problem upon which we build a general surface area formula. First consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0, h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the x-axis, it generates the surface of a cone of radius r and height h (**Figure 6.60**). A formula from geometry states that the surface area of a right circular cone of radius r and height h (excluding the base) is $\pi r\sqrt{r^2+h^2} = \pi r \ell$, where ℓ is the slant height of the cone (the length of the slanted "edge" of the cone).

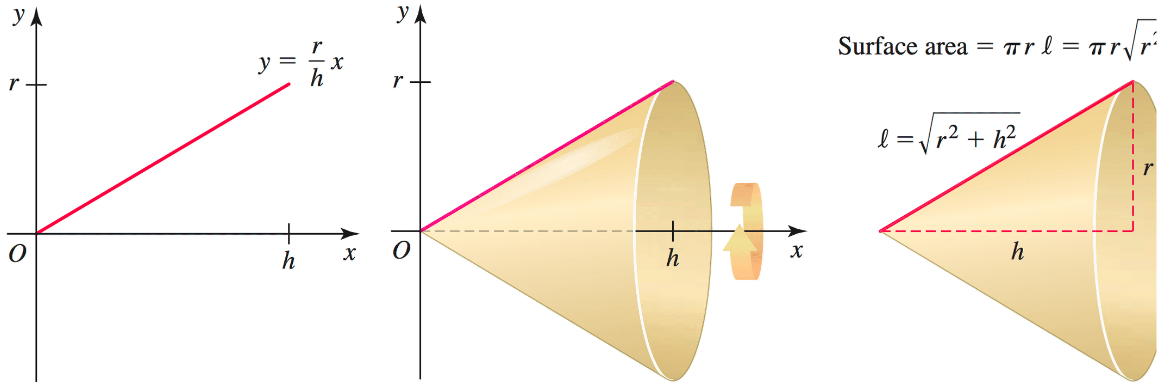


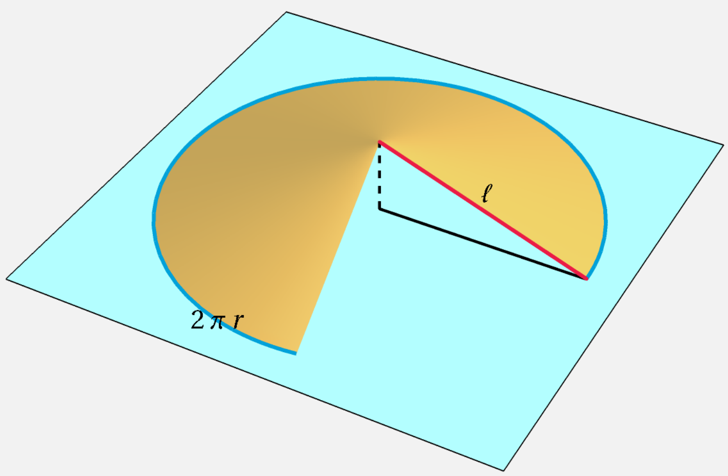
Figure 6.60

Note »

One way to derive the formula for the surface area of a cone is to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ with a curved edge of length $2\pi r$. This sector is a fraction $\frac{2\pi r}{2\pi \ell} = \frac{r}{\ell}$ of a full circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi \ell^2 \cdot \frac{r}{\ell} = \pi r \ell$.

unfold the cone

Curved edge length $= 2\pi r$



Quick Check 1 Which is greater, the surface area of a cone of height 10 and radius 20 or the surface area of a cone of height 20 and radius 10 (excluding the bases)? ◆

Answer »

The surface area of the first cone ($200\sqrt{5}\pi$) is twice as great as the surface area of the second cone ($100\sqrt{5}\pi$).

With this result, we can solve a preliminary problem that will be useful. Consider the linear function $f(x) = cx$ on the interval $[a, b]$, where $0 < a < b$ and $c > 0$. When this line segment is revolved about the x-axis, it generates a *frustum of a cone* (a cone whose top has been sliced off). The goal is to find S, the surface area of the frustum. **Figure 6.61** shows that S is the difference between the surface area S_b of the cone that extends over the interval $[0, b]$ and the surface area S_a of the cone that extends over the interval $[0, a]$.

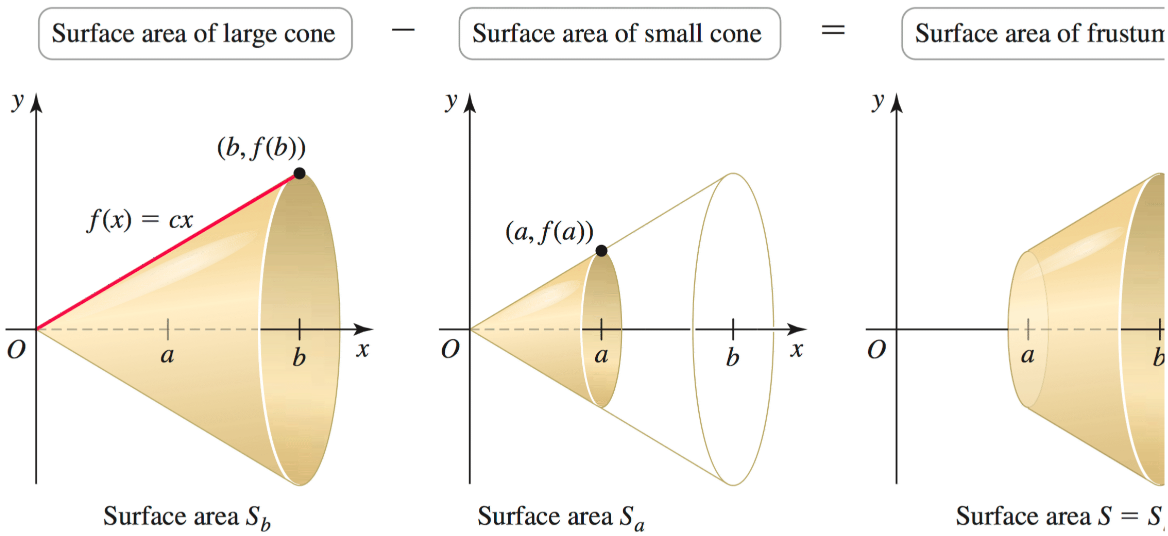


Figure 6.61

Notice that the radius of the cone on $[0, b]$ is $r = f(b) = c\,b$, and its height is $h = b$. Therefore, this cone has surface area

$$S_b = \pi\, r\, \sqrt{r^2 + h^2} = \pi\,(b\,c)\,\sqrt{(b\,c)^2 + b^2} = \pi\, b^2\, c\, \sqrt{c^2 + 1}\,.$$

Similarly, the cone on $[0, a]$ has radius $r = f(a) = c\,a$ and height $h = a$, so its surface area is

$$S_a = \pi\,(a\,c)\,\sqrt{(a\,c)^2 + a^2} = \pi\, a^2\, c\, \sqrt{c^2 + 1}\,.$$

The difference of the surface areas $S_b - S_a$ is the surface area S of the frustum on $[a, b]$:

$$\begin{aligned} S &= S_b - S_a \\ &= \pi\, b^2\, c\, \sqrt{c^2 + 1} - \pi\, a^2\, c\, \sqrt{c^2 + 1} \\ &= \pi\, c\left(b^2 - a^2\right)\sqrt{c^2 + 1}\,. \end{aligned}$$

A slightly different form of this surface area formula will be useful. Observe that the line segment between $(a,\ f(a))$ and $(b,\ f(b))$ (which is the slant height of the frustum in Figure 6.61) has length

$$\ell = \sqrt{(b - a)^2 + (b\,c - a\,c)^2} = (b - a)\,\sqrt{c^2 + 1}\,.$$

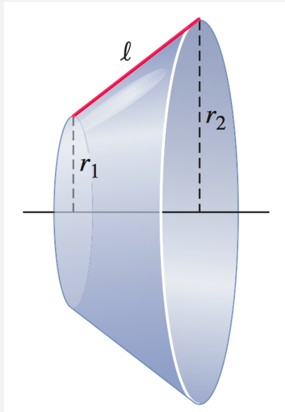
Therefore, the surface area of the frustum can also be written

$$\begin{aligned} S &= \pi\, c\left(b^2 - a^2\right)\sqrt{c^2 + 1} \\ &= \pi\, c\,(b + a)\,(b - a)\,\sqrt{c^2 + 1} && \text{Factor } b^2 - a^2. \\ &= \pi\left(\underbrace{c\,b}_{f(b)} + \underbrace{c\,a}_{f(a)}\right)\underbrace{(b - a)\,\sqrt{c^2 + 1}}_{\ell} && \text{Expand } c\,(b + a). \\ &= \pi\,(f(b) + f(a))\,\ell. \end{aligned}$$

Note »

Surface area of frustum:

$$\begin{aligned} S &= \pi\,(f(b) + f(a))\,\ell \\ &= \pi\,(r_2 + r_1)\,\ell \end{aligned}$$



This result can be generalized to *any* linear function $g(x) = c\,x + d$ that is positive on the interval $[a, b]$. That is, the surface area of the frustum generated by revolving the line segment between $(a, g(a))$ and $(b, g(b))$ about the x-axis is given by $\pi\,(g(b) + g(a))\,\ell$ (Exercise 38).

Quick Check 2 What is the surface area of the frustum of a cone generated when the graph of $f(x) = 3\,x$ on the interval $[2, 5]$ is revolved about the x-axis? ◆

Answer »

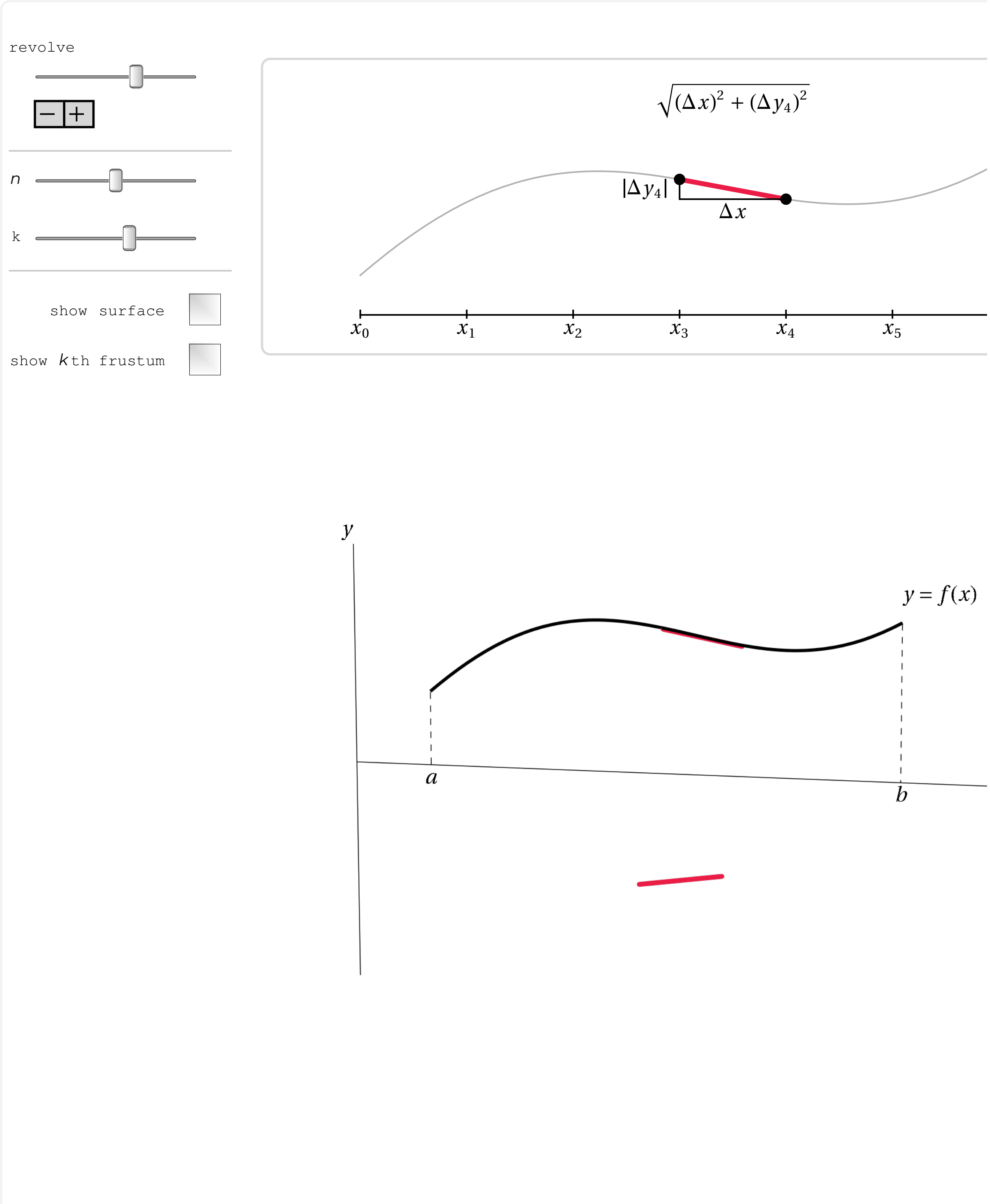
The surface area is $63\sqrt{10}\,\pi$.

Surface Area Formula »

With the surface area formula for a frustum of a cone in hand, we now derive a general area formula for a surface of revolution. We assume the surface is generated by revolving the graph of a positive, differentiable function f on the interval $[a, b]$ about the x-axis. We begin by subdividing the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b - a}{n}$. The grid points in this partition are

$$x_0 = a,\; x_1,\; x_2,\; \ldots,\; x_{n-1},\; x_n = b.$$

Now consider the kth subinterval $[x_{k-1}, x_k]$ and the line segment between the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ (**Figure 6.62**). We let the change in the y-coordinates between these points be $\Delta y_k = f(x_k) - f(x_{k-1})$.



Figures 6.62 and 6.63

When this line segment is revolved about the x-axis, it generates a frustum of a cone (**Figure 6.63**). The slant height of this frustum is the length of the hypotenuse of a right triangle whose sides have lengths Δx and $|\Delta y_k|$. Therefore, the slant height of the kth frustum is

$$\sqrt{(\Delta x)^2 + |\Delta y_k|^2} = \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

and its surface area is

$$S_k = \pi\,(f(x_k) + f(x_{k-1}))\,\sqrt{(\Delta x)^2 + (\Delta y_k)^2}\,.$$

It follows that the area S of the entire surface of revolution is approximately the sum of the surface areas of the individual frustums S_k, for $k = 1, \ldots, n$; that is,

$$S \approx \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} \pi\,(f(x_k) + f(x_{k-1}))\,\sqrt{(\Delta x)^2 + (\Delta y_k)^2}\,.$$

We would like to identity this sum as a Riemann sum. However, one more step is required to put it in the correct form. We apply the Mean Value Theorem on the kth subinterval $[x_{k-1}, x_k]$ and observe that

$$\frac{f(x_k) - f(x_{k-1})}{\Delta x} = f'(x_k^*),$$

for some number x_k^* in the interval (x_{k-1}, x_k), for $k = 1, \ldots, n$. It follows that $\Delta y_k = f(x_k) - f(x_{k-1}) = f'(x_k^*)\,\Delta x$.

Note »

Notice that f is assumed to be differentiable on $[a, b]$; therefore, it satisfies the conditions of the Mean Value Theorem. Recall that a similar argument was used to derive the arc length formula in Section 6.5.

We now replace Δy_k by $f'(x_k^*)\,\Delta x$ in the expression for the approximate surface area. The result is

$$\begin{aligned} S \approx \sum_{k=1}^{n} S_k &= \sum_{k=1}^{n} \pi\,(f(x_k) + f(x_{k-1}))\,\sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\ &= \sum_{k=1}^{n} \pi\,(f(x_k) + f(x_{k-1}))\,\sqrt{(\Delta x)^2\left(1 + f'(x_k^*)^2\right)} \quad \text{Mean Value Theorem} \\ &= \sum_{k=1}^{n} \pi\,(f(x_k) + f(x_{k-1}))\,\sqrt{1 + f'(x_k^*)^2}\;\Delta x. \quad \text{Factor out } \Delta x. \end{aligned}$$

When Δx is small, we have $x_{k-1} \approx x_k \approx x_k^*$, and by the continuity of f, it follows that $f(x_{k-1}) \approx f(x_k) \approx f(x_k^*)$, for $k = 1, \ldots, n$. These observations allow us to write

$$\begin{aligned} S &\approx \sum_{k=1}^{n} \pi\,(f(x_k^*) + f(x_k^*))\,\sqrt{1 + f'(x_k^*)^2}\;\Delta x \\ &= \sum_{k=1}^{n} 2\pi\, f(x_k^*)\,\sqrt{1 + f'(x_k^*)^2}\;\Delta x. \end{aligned}$$

This approximation to S, which has the form of a Riemann sum, improves as the number of subintervals increases and as the length of the subintervals approaches 0. Specifically, as $n \to \infty$ and as $\Delta x \to 0$, we obtain an integral for the surface area:

$$S = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + f'(x_k^*)^2}\, \Delta x$$
$$= \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx.$$

DEFINITION **Area of a Surface of Revolution**

Let f be differentiable and positive on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx.$$

Quick Check 3 Let $f(x) = c$, where $c > 0$. What surface is generated when the graph of f on $[a, b]$ is revolved about the x-axis? Without using calculus, what is the area of the surface? ◆

Answer »

The surface is a cylinder of radius c and height $b - a$. The area of the curved surface is $2\pi c(b - a)$.

EXAMPLE 1 Using the surface area formula

The graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of the surface generated (**Figure 6.64**)?

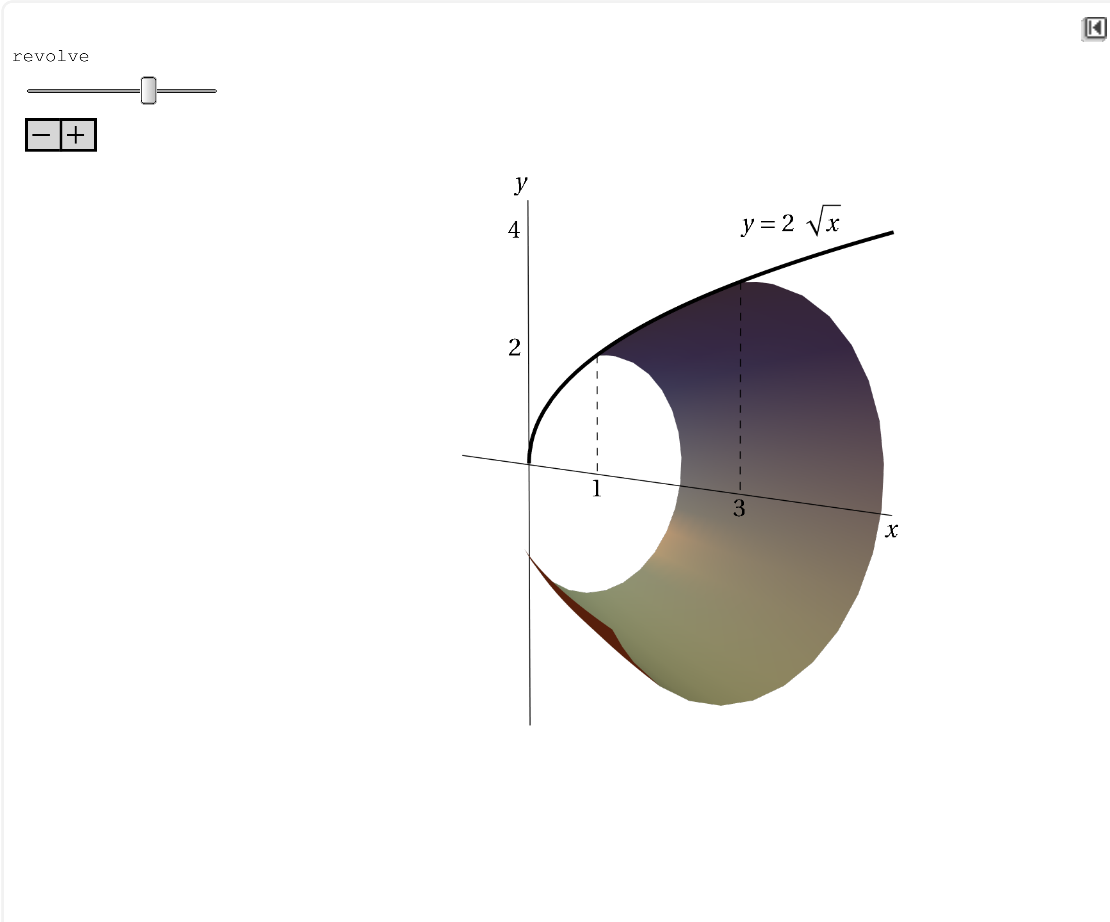


Figure 6.64

SOLUTION »

Noting that $f'(x) = \frac{1}{\sqrt{x}}$, the surface area formula gives

$$\begin{aligned} S &= \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx \\ &= 2\pi\int_1^3 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\,dx && \text{Substitute for } f \text{ and } f'. \\ &= 4\pi\int_1^3 \sqrt{x+1}\,dx && \text{Simplify.} \\ &= \frac{8\pi}{3}(x+1)^{3/2}\Big|_1^3 = \frac{16\pi}{3}\left(4 - \sqrt{2}\right). && \text{Integrate and simplify.} \end{aligned}$$

Related Exercise 9 ◆

EXAMPLE 2 Surface area of a spherical zone

A spherical zone is produced when a sphere of radius a is sliced by two parallel planes. In this example, we compute the surface area of the spherical zone that results when the first plane is oriented vertically and cuts the sphere in half while the second plane lies h units to the right, where $0 \le h < a$ (**Figure 6.65**). Show that the area of this spherical zone of width h cut from a sphere of radius a is $2\pi a h$, and use the result to show that the surface area of the sphere is $4\pi a^2$.

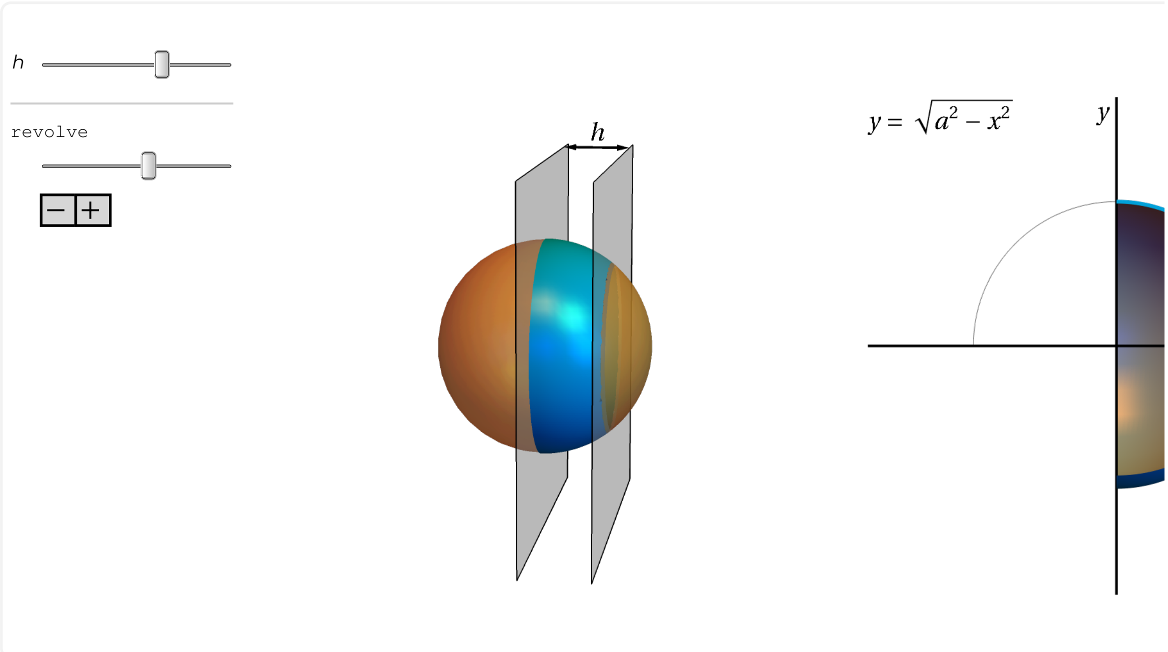


Figure 6.65

SOLUTION »

To generate the spherical zone, we revolve the curve $f(x) = \sqrt{a^2 - x^2}$ on the interval $[0, h]$ about the x-axis (Figure 6.65). Noting that $f'(x) = -x\left(a^2 - x^2\right)^{-1/2}$, the surface area of the spherical zone of width h is

$$
\begin{aligned}
S &= \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \\
&= 2\pi \int_0^h \sqrt{a^2 - x^2}\sqrt{1 + \left(-x\left(a^2 - x^2\right)^{-1/2}\right)^2}\, dx && \text{Substitute for } f \text{ and } f'. \\
&= 2\pi \int_0^h \sqrt{a^2 - x^2}\sqrt{\frac{a^2}{a^2 - x^2}}\, dx && \text{Simplify.} \\
&= 2\pi \int_0^h a\, dx = 2\pi a h. && \text{Simplify and integrate.}
\end{aligned}
$$

To find the surface area of the entire sphere, it is tempting to integrate over the interval $[-a, a]$. However, f is not differentiable at $\pm a$, so we cannot use the surface area formula. Instead, we use the formula for the area of the spherical zone and let h approach a from the left to find the area of the associated hemisphere:

$$
\begin{aligned}
\text{Area of zone} &= 2\pi a h, \text{ so} \\
\text{Area of hemisphere} &= \lim_{h \to a^-} 2\pi a h = 2\pi a^2.
\end{aligned}
$$

Therefore, the surface area of a sphere of radius a is $2\left(2\pi a^2\right) = 4\pi a^2$.

Related Exercise 13 ◆

EXAMPLE 3 **Painting a funnel**

The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval [1, 2] about the x-axis (**Figure 6.66**). Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 cm thick? Assume that x and y are measured in centimeters.

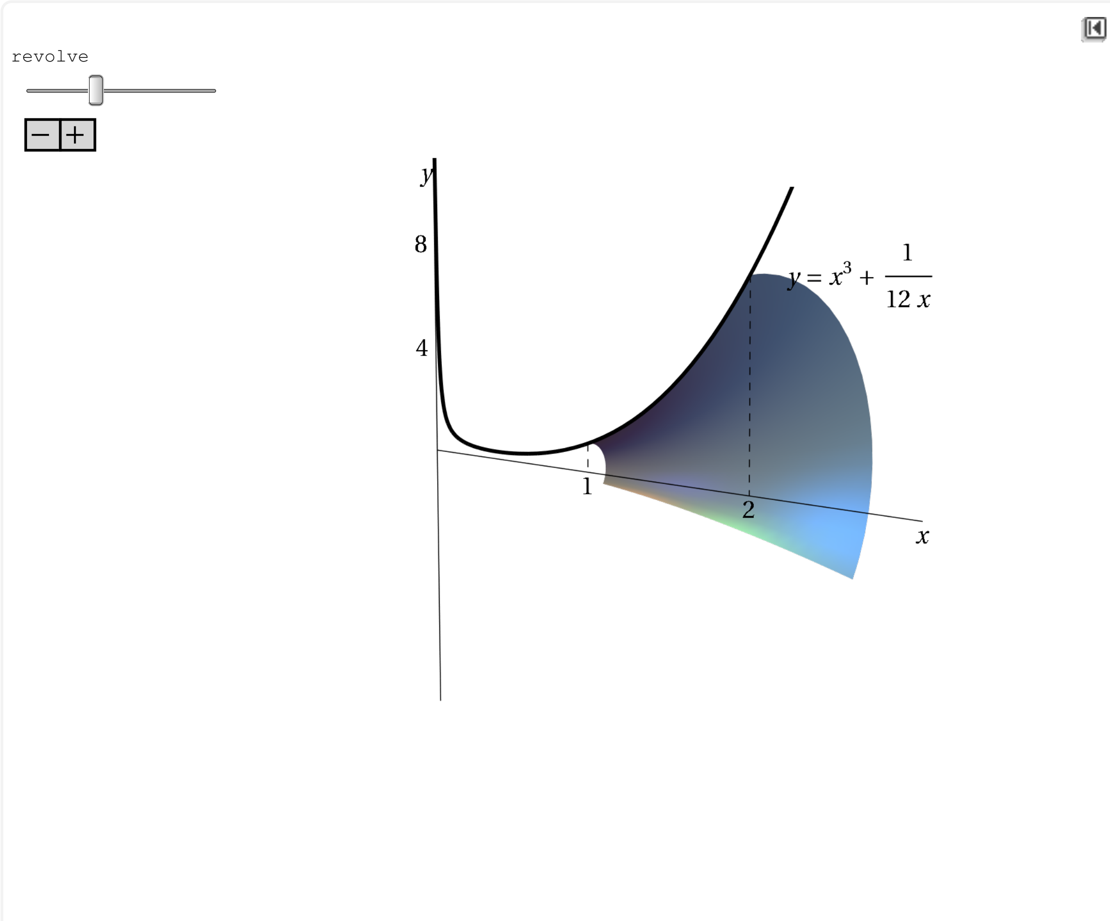


Figure 6.66

SOLUTION »

Note that $f'(x) = 3x^2 - \frac{1}{12x^2}$. Therefore, the surface area of the funnel in cm^2 is

$$
\begin{aligned}
S &= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \\
&= 2\pi\int_1^2\left(x^3+\frac{1}{12x}\right)\sqrt{1+\left(3x^2-\frac{1}{12x^2}\right)^2}\,dx && \text{Substitute for } f \text{ and } f'. \\
&= 2\pi\int_1^2\left(x^3+\frac{1}{12x}\right)\sqrt{\left(3x^2+\frac{1}{12x^2}\right)^2}\,dx && \text{Expand and factor under square root.} \\
&= 2\pi\int_1^2\left(x^3+\frac{1}{12x}\right)\left(3x^2+\frac{1}{12x^2}\right)dx && \text{Simplify.} \\
&= \frac{12{,}289}{192}\pi. && \text{Evaluate integral.}
\end{aligned}
$$

Because the paint layer is 0.05 cm thick, the volume of paint needed is approximately

$$\left(\frac{12{,}289\,\pi}{192}\text{ cm}^2\right)(0.05\text{ cm}) \approx 10.1\text{ cm}^3.$$

Related Exercises 21–22 ◆

The derivation that led to the surface area integral may be used when a curve is revolved about the y-axis (rather than the x-axis). The result is the same integral with x replaced by y. For example, if the curve $x = g(y)$ on the interval $[c, d]$ is revolved about the y-axis, the area of the surface generated is

$$S = \int_c^d 2\pi\, g(y)\sqrt{1+g'(y)^2}\,dy.$$

To use this integral, we must first describe the given curve as a function of y.

EXAMPLE 4 Change of perspective

Consider the curve defined by the equation $4y^3 - 10x\sqrt{y} + 5 = 0$, for $1 \le y \le 2$. Find the area of the surface generated when this curve is revolved about the y-axis (**Figure 6.67**).

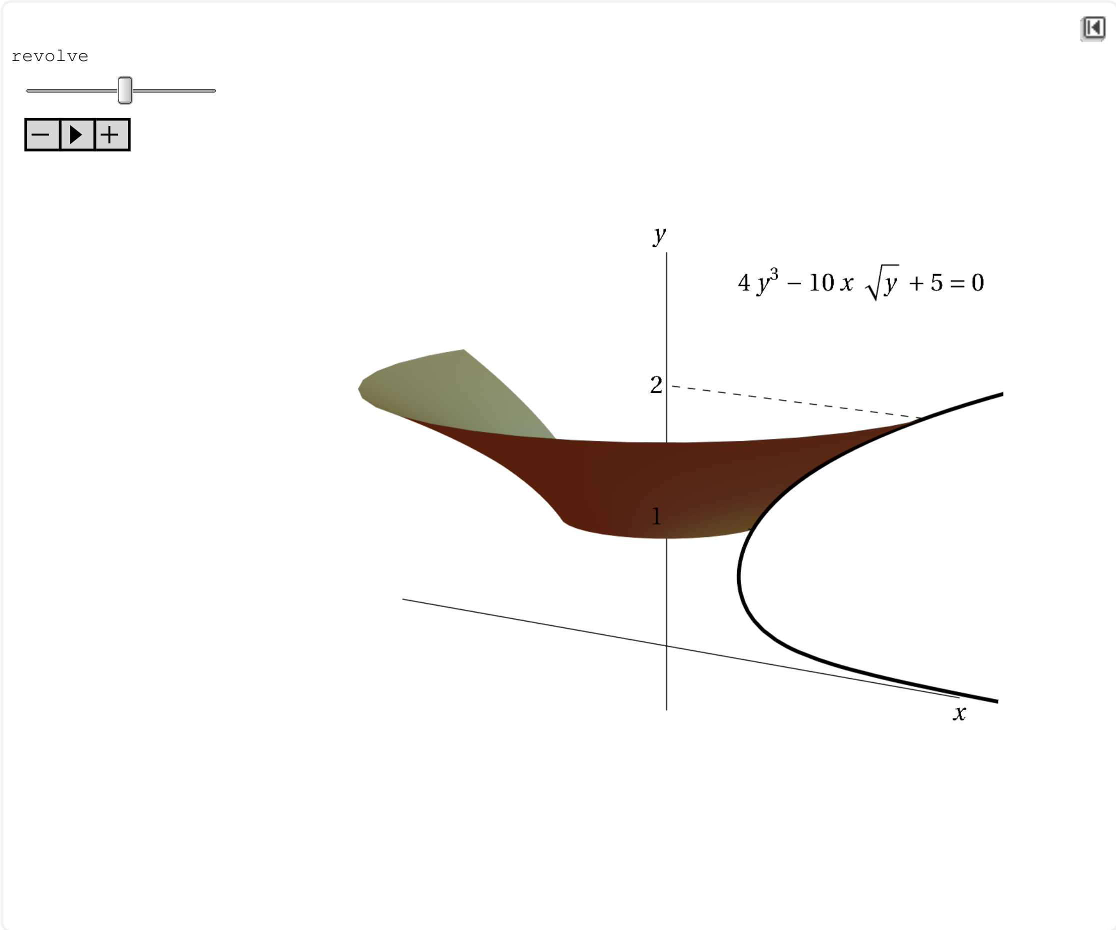


Figure 6.67

SOLUTION

We solve the equation $4y^3 - 10x\sqrt{y} + 5 = 0$ for x so that the curve is expressed as a function of y:

$$4y^3 - 10x\sqrt{y} + 5 = 0$$

$$10x\sqrt{y} = 4y^3 + 5 \qquad \text{Rearrange terms.}$$

$$x = g(y) = \frac{2}{5}y^{5/2} + \frac{1}{2}y^{-1/2}. \qquad \text{Divide by } 10\sqrt{y}.$$

Note that $g'(y) = y^{3/2} - \frac{1}{4}y^{-3/2}$ and that the interval of integration on the y-axis is $[1, 2]$. The area of the surface is

$$S = \int_c^d 2\pi\, g(y)\, \sqrt{1 + g'(y)^2}\; dy$$

$$= 2\pi \int_1^2 \left(\frac{2}{5} y^{5/2} + \frac{1}{2} y^{-1/2}\right) \sqrt{1 + \left(y^{3/2} - \frac{1}{4} y^{-3/2}\right)^2}\; dy \quad \text{Substitute for } g \text{ and } g'.$$

$$= 2\pi \int_1^2 \left(\frac{2}{5} y^{5/2} + \frac{1}{2} y^{-1/2}\right) \sqrt{\left(y^{3/2} + \frac{1}{4} y^{-3/2}\right)^2}\; dy \quad \text{Expand and factor.}$$

$$= 2\pi \int_1^2 \left(\frac{2}{5} y^4 + \frac{3}{5} y + \frac{1}{8} y^{-2}\right) dy \quad \text{Simplify.}$$

$$= 2\pi \left(\frac{2}{25} y^5 + \frac{3}{10} y^2 - \frac{1}{8y}\right)\Bigg|_1^2 = \frac{1377}{200}\pi. \quad \text{Integrate and evaluate.}$$

Related Exercises 34 ◆

Exercises »

Getting Started »

Practice Exercises »

7–20. Computing surface areas *Find the area of the surface generated when the given curve is revolved about the given axis.*

7. $y = 3x + 4$, for $0 \le x \le 6$; about the x-axis

8. $y = 12 - 3x$, for $1 \le x \le 3$; about the x-axis

9. $y = 8\sqrt{x}$, for $9 \le x \le 20$; about the x-axis

10. $y = x^3$, for $0 \le x \le 1$; about the x-axis

11. $y = (3x)^{1/3}$, for $0 \le x \le \frac{8}{3}$; about the y-axis

12. $y = \frac{x^2}{4}$, for $2 \le x \le 4$; about the y-axis

13. $y = \sqrt{1 - x^2}$, for $-\frac{1}{2} \le x \le \frac{1}{2}$; about the x-axis

14. $x = \sqrt{-y^2 + 6y - 8}$, for $3 \le y \le \frac{7}{2}$; about the y-axis

15. $y = 4x - 1$, for $1 \le x \le 4$; about the y-axis (Hint: Integrate with respect to y.)

16. $y = \sqrt{4x + 6}$, for $0 \le x \le 5$; about the x-axis

17. $y = \sqrt{-x^2 + 6x - 5}$, for $2 \le x \le 3$; about the x-axis

18. $y = \sqrt{5x - x^2}$, for $1 \le x \le 4$; about the x-axis

19. $x = \sqrt{12y - y^2}$, for $2 \le y \le 10$; about the y-axis

20. $y = 1 + \sqrt{1 - x^2}$ between the points $(1, 1)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$; about the y-axis

21–22. Painting surfaces *A* 1.5-mm *layer of paint is applied to one side of the following surfaces. Find the approximate volume of paint needed. Assume x and y are measured in meters.*

21. The spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $1 \le x \le 7$ is revolved about the x-axis

22. The spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $-8 \le x \le 8$ is revolved about the x-axis

23. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If the curve $y = f(x)$ on the interval $[a, b]$ is revolved about the y-axis, the area of the surface generated is

$$\int_{f(a)}^{f(b)} 2\pi f(y) \sqrt{1 + f'(y)^2}\, dy.$$

b. If f is not one-to-one on the interval $[a, b]$, then the area of the surface generated when the graph of f on $[a, b]$ is revolved about the x-axis is not defined.

c. Let $f(x) = 12x^2$. The area of the surface generated when the graph of f on $[-4, 4]$ is revolved about the x-axis is twice the area of the surface generated when the graph of f on $[0, 4]$ is revolved about the x-axis.

d. Let $f(x) = 12x^2$. The area of the surface generated when the graph of f on $[-4, 4]$ is revolved about the y-axis is twice the area of the surface generated when the graph of f on $[0, 4]$ is revolved about the y-axis.

T **24–28. Surface area using technology** *Consider the following curves on the given intervals.*

a. *Write the integral that gives the area of the surface generated when the curve is revolved about the given axis.*

b. *Use a calculator or software to approximate the surface area.*

24. $y = x^5$, for $0 \le x \le 1$; about the x-axis

25. $y = \cos x$, for $0 \le x \le \frac{\pi}{2}$; about the x-axis

26. $y = (2x + 3)^2$, for $0 \le x \le 1$; about the y-axis

27. $y = \tan x$, for $0 \le x \le \frac{\pi}{4}$; about the x-axis

28. $y = \sqrt{\sin \pi x}$, for $0 \le x \le \frac{1}{2}$; about the x-axis

29. Revolving an astroid Consider the upper half of the astroid described by $x^{2/3} + y^{2/3} = a^{2/3}$, where $a > 0$ and $|x| \le a$. Find the area of the surface generated when this curve is revolved about the x-axis. Note that the function describing the curve is not differentiable at 0. However, the surface area integral can be evaluated using symmetry and methods you know.

30. Cones and cylinders The volume of a cone of radius r and height h is one-third the volume of a cylinder with the same radius and height. Does the surface area of a cone of radius r and height h equal one-third the surface area of a cylinder with the same radius and height? If not, find the correct relationship. Exclude the bases of the cone and cylinder.

Explorations and Challenges »

31–35. Challenging surface area calculations *Find the area of the surface generated when the given curve is revolved about the given axis*

31. $y = x^{3/2} - \dfrac{x^{1/2}}{3}$, for $1 \le x \le 2$; about the x-axis

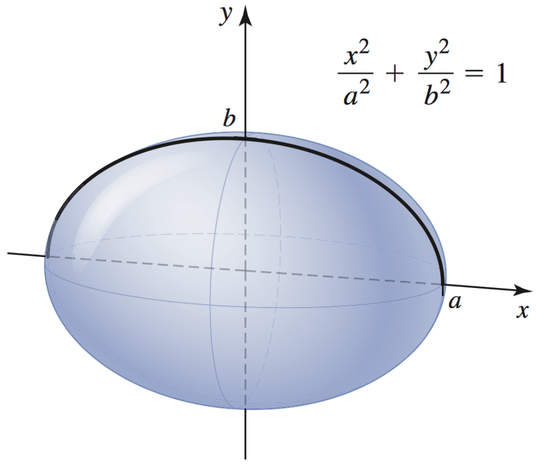
32. $y = \dfrac{x^4}{8} + \dfrac{1}{4\,x^2}$, for $1 \le x \le 2$; about the x-axis

33. $y = \dfrac{x^3}{3} + \dfrac{1}{4\,x}$, for $\dfrac{1}{2} \le x \le 2$; about the x-axis

34. $y = \dfrac{1}{2}\left(1 - \sqrt{1 - 4\,x^2}\right)$ between the points $\left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ and $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$; about the y-axis

35. $x = 4\,y^{3/2} - \dfrac{y^{1/2}}{12}$, for $1 \le y \le 4$; about the y-axis

36. Surface area of an ellipsoid If the top half of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is revolved about the x-axis, the result is an *ellipsoid* whose axis along the x-axis has length $2\,a$, whose axis along the y-axis has length $2\,b$, and whose axis perpendicular to the xy-plane has length $2\,b$. We assume that $0 < b < a$ (see figure). Use the following steps to find the surface area S of this ellipsoid.



a. Use the surface area formula to show that $S = \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - c^2x^2}\,dx$, where $c^2 = 1 - \frac{b^2}{a^2}$.

b. Use the change of variables $u = cx$ to show that $S = \frac{4\pi b}{\sqrt{a^2 - b^2}}\int_0^{\sqrt{a^2-b^2}} \sqrt{a^2 - u^2}\,du$.

c. **T** Use the integral formula in part (b) together with a calculator or software to approximate the surface area of an ellipse if $a = 4$ and $b = 3$.

d. **T** The surface area of the ellipsoid can be estimated by the Knud Thomsen formula

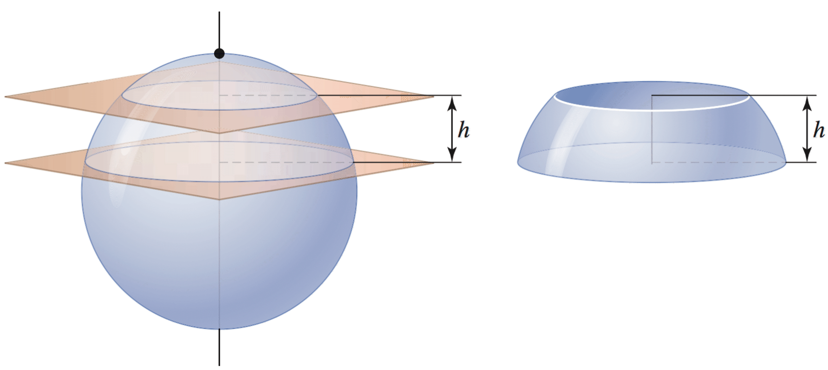
$$S \approx 4\pi\left(\frac{2(ab)^p + b^{2p}}{3}\right)^{1/p},$$

where $p = 1.6075$. Use this formula to estimate the surface area of an ellipse, where $a = 4$ and $b = 3$. Compare this approximation with the result found in part (c).

e. Use part (a) to show that if $a = b$, then $S = 4\pi a^2$, which is the surface area of a sphere of radius a.

f. How accurate is the formula in part (d) if $a = b$?

37. **Zones of a sphere** Suppose a sphere of radius r is sliced by two horizontal planes h units apart (see figure). Show that the surface area of the resulting zone on the sphere is $2\pi rh$, independent of the location of the cutting planes.



38. Surface area of a frustum Show that the surface area of the frustum of a cone generated by revolving the line segment between $(a, g(a))$ and $(b, g(b))$ about the x-axis is $\pi(g(b) + g(a))\ell$, for any linear function $g(x) = cx + d$ that is positive on the interval $[a, b]$, where ℓ is the slant height of the frustum.

39. Scaling surface area Let f be a nonnegative function with a continuous first derivative on $[a, b]$ and suppose $g(x) = c f(x)$ and $h(x) = f(cx)$, where $c > 0$. When the curve $y = f(x)$ on $[a, b]$ is revolved about the x-axis, the area of the resulting surface is A. Evaluate the following integrals in terms of A and c.

a. $\displaystyle\int_a^b 2\pi g(x)\sqrt{c^2 + g'(x)^2}\,dx$

b. $\displaystyle\int_{a/c}^{b/c} 2\pi h(x)\sqrt{c^2 + h'(x)^2}\,dx$

40. Surface plus cylinder Suppose f is a nonnegative function with a continuous first derivative on $[a, b]$. Let L equal the length of the graph of f on $[a, b]$ and let S be the area of the surface generated by revolving the graph of f on $[a, b]$ about the x-axis. For a positive constant C, assume the curve $y = f(x) + C$ is revolved about the x-axis. Show that the area of the resulting surface equals the sum of the S and the surface area of a right circular cylinder of radius C and height L.

41. Surface-area-to-volume ratio (SAV) In the design of solid objects (both artificial and natural), the ratio of the surface area to the volume of the object is important. Animals typically generate heat at a rate proportional to their volume and lose heat at a rate proportional to their surface area. Therefore, animals with a low SAV ratio tend to retain heat whereas animals with a high SAV ratio (such as children and hummingbirds) lose heat relatively quickly.

a. What is the SAV ratio of a cube with side lengths R?

b. What is the SAV ratio of a ball with radius R?

c. For a fixed constant $h > 0$, consider a ball of radius $\sqrt[3]{\frac{h}{4}}$ and an ellipsoid with a long axis of length $2\sqrt[3]{h}$ and short axes that are each half the length of the long axis. Show that the ball and ellipsoid have the same volume. (*Hint:* The volume of an ellipsoid is $\frac{4}{3}\pi ABC$, where the axes have lengths $2A$, $2B$, and $2C$.)

d. **T** Create a table to compare SAV ratios of the balls and ellipsoids of equal volume described in part (c) using values of $h = 1.1$, 5, 10, and 20. (*Hint:* Find the surface area of the ellipsoids using the integral in part (b) of Exercise 36 together with a calculator or other technology.)

e. Among all ellipsoids of a fixed volume, which one would you choose for the shape of an animal if the goal is to minimize heat loss?