6.5 Length of Curves

The space station orbits Earth in an elliptical path. How far does it travel in one orbit? A baseball slugger launches a home run into the upper deck and the sportscaster claims it landed 480 feet from home plate. But how far did the ball actually travel along its flight path? These questions deal with the length of trajectories or, more generally, with *arc length*. As you will see, their answers can be found by integration.

There are two common ways to formulate problems about arc length: The curve may be given explicitly in the form $y = f(x)$ or it may be defined *parametrically*. In this section we deal with the first case. Parametric curves and the associated arc length problem are discussed in Section 12.1.

Arc Length for $y = f(x)$ »

Suppose a curve is given by $y = f(x)$, where f is a function with a continuous first derivative on the interval $[a, b]$. The goal is to determine how far you would travel if you walked along the curve from $(a, f(a))$ to $(b, f(b))$. This distance is the arc length, which we denote L.

As shown in Figure 6.55, we divide $[a, b]$ into n subintervals of length $\Delta x = \dfrac{b-a}{n}$, where x_k is the right endpoint of the kth subinterval, for $k = 1, \ldots, n$. Joining the corresponding points on the curve by line segments, we obtain a polygonal line with n line segments. If n is large and Δx is small, the length of the polygonal line is a good approximation to the length of the actual curve. The strategy is to find the length of the polygonal line and then let n increase, while Δx goes to zero, to get the exact length of the curve.

Note »

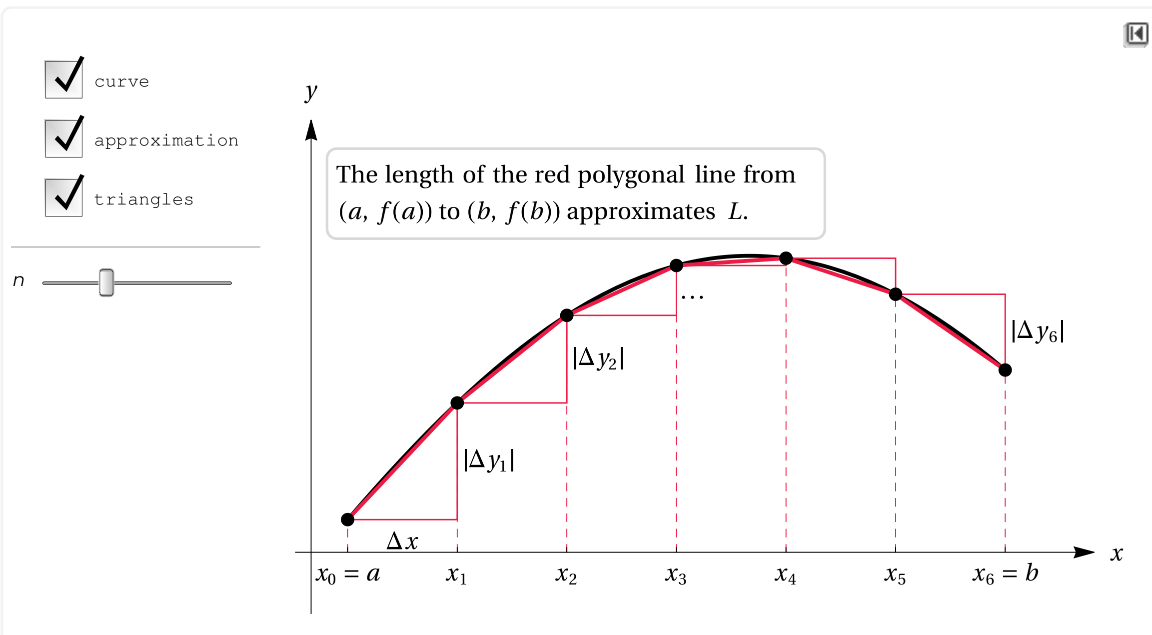


Figure 6.55

Consider the kth subinterval $[x_{k-1}, x_k]$ and the line segment between the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. We let the change in the y-coordinate between these points be

$$\Delta y_k = f(x_k) - f(x_{k-1}).$$

The kth line is the hypotenuse of a right triangle with sides of length Δx and $|\Delta y_k| = |f(x_k) - f(x_{k-1})|$. The length of each line segment is

$$\sqrt{(\Delta x)^2 + |\Delta y_k|^2}, \quad \text{for } k = 1, 2, \ldots, n.$$

Summing these lengths, we obtain the length of the polygonal line, which approximates the length L of the curve:

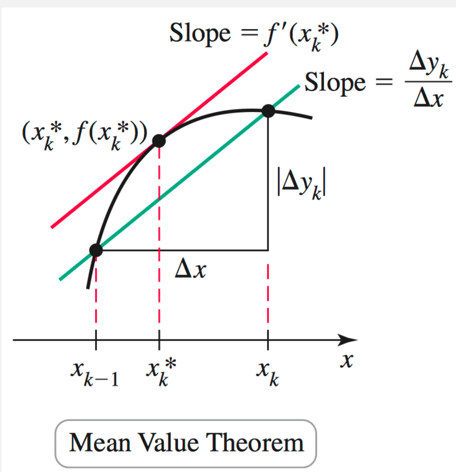
$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + |\Delta y_k|^2}.$$

Note »

Notice that Δx is the same for each subinterval, but Δy_k depends on the subinterval.

In previous applications of the integral, we would, at this point, take the limit as $n \to \infty$ and $\Delta x \to 0$ to obtain a definite integral. However, because of the presence of the Δy_k term, we must complete one additional step before taking a limit. Notice that the slope of the line segment on the kth subinterval is $\dfrac{\Delta y_k}{\Delta x}$ (rise over run). By the Mean Value Theorem (see Note and Section 4.2), this slope equals $f'(x_k^*)$, for some point x_k^* on the kth subinterval.

Note »



Therefore,

$$\begin{aligned} L &\approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + |\Delta y_k|^2} \\ &= \sum_{k=1}^{n} \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2\right)} && \text{Factor out } (\Delta x)^2. \\ &= \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2}\ \Delta x && \text{Bring } \Delta x \text{ out of the square root.} \\ &= \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)^2}\ \Delta x. && \text{Mean Value Theorem} \end{aligned}$$

Now we have a Riemann sum. As n increases and as Δx approaches zero, the sum approaches a definite integral, which is also the exact length of the curve. We have

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + f'(x_k^*)^2}\ \Delta x = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Note »

Note that $1 + f'(x)^2$ is positive, so the square root in the integrand is defined whenever f' exists. To ensure that $\sqrt{1 + f'(x)^2}$ is integrable on $[a, b]$, we require that f' be continuous on $[a, b]$.

DEFINITION **Arc Length for $y = f(x)$**

Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Quick Check 1 What does the arc length formula give for the length of the line $y = x$ between $x = 0$ and $x = a$, where $a \ge 0$? ◆

Answer »

$\sqrt{2}\ a$ (the length of the line segment joining the points)

EXAMPLE 1 Arc length

Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$ (**Figure 6.56**).

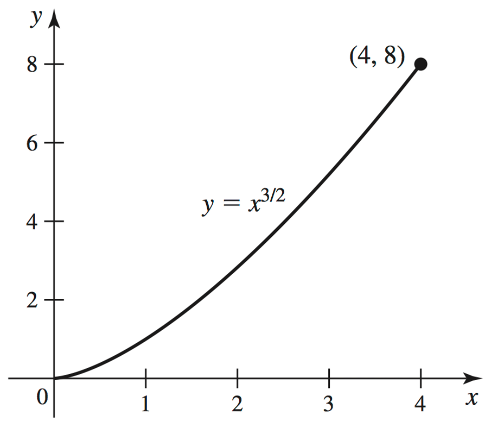


Figure 6.56

SOLUTION »

Notice that $f'(x) = \frac{3}{2}x^{1/2}$, which is continuous on $[0, 4]$. Using the arc length formula, we have

$$\begin{aligned} L = \int_a^b \sqrt{1 + f'(x)^2}\, dx &= \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx && \text{Substitute for } f'(x). \\ &= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx && \text{Simplify.} \\ &= \frac{4}{9}\int_1^{10} \sqrt{u}\, du && u = 1 + \frac{9x}{4},\ du = \frac{9}{4}dx \\ &= \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{10} && \text{Fundamental Theorem} \\ &= \frac{8}{27}\left(10^{3/2} - 1\right). && \text{Simplify.} \end{aligned}$$

The length of the curve is $\frac{8}{27}\left(10^{3/2} - 1\right) \approx 9.1$ units.

Related Exercise 11 ◆

EXAMPLE 2 Arc length calculation

Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ on the interval $\left[\frac{1}{2}, 2\right]$ (**Figure 6.57**).

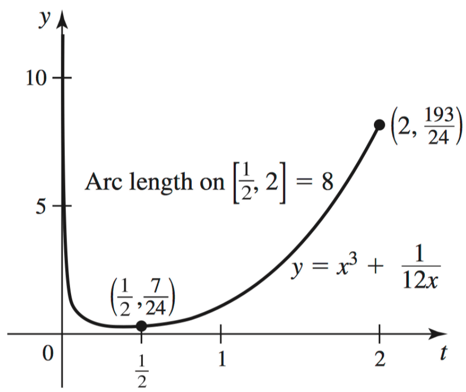


Figure 6.57

SOLUTION

We first calculate $f'(x) = 3x^2 - \dfrac{1}{12x^2}$ and $f'(x)^2 = 9x^4 - \dfrac{1}{2} + \dfrac{1}{144x^4}$. The length of the curve on $\left[\dfrac{1}{2}, 2\right]$ is

$$\begin{aligned} L &= \int_{1/2}^{2} \sqrt{1 + f'(x)^2}\, dx \\ &= \int_{1/2}^{2} \sqrt{1 + \left(9x^4 - \frac{1}{2} + \frac{1}{144x^4}\right)}\, dx && \text{Substitute.} \\ &= \int_{1/2}^{2} \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx && \text{Factor.} \\ &= \int_{1/2}^{2} \left(3x^2 + \frac{1}{12x^2}\right) dx && \text{Simplify.} \\ &= \left(x^3 - \frac{1}{12x}\right)\Bigg|_{1/2}^{2} = 8. && \text{Evaluate the integral.} \end{aligned}$$

Related Exercise 10 ◆

EXAMPLE 3 Looking ahead

Consider the segment of the parabola $f(x) = x^2$ on the interval $[0, 2]$.

a. Write the integral for the length of the curve.

b. Use a calculator to evaluate the integral.

SOLUTION »

a. Noting that $f'(x) = 2x$, the arc length integral is

$$\int_0^2 \sqrt{1 + f'(x)^2}\, dx = \int_0^2 \sqrt{1 + 4x^2}\, dx.$$

b. Using integration techniques presented so far, this integral cannot be evaluated (the required method is given in Section 8.4). This is typical of arc length integrals—even simple functions can lead to arc length integrals that are difficult to evaluate analytically. Without an analytical method, we may use numerical integration to *approximate* the value of a definite integral (Section 8.8). Many calculators have built-in functions for this purpose. For this integral, the approximate arc length is

$$\int_0^2 \sqrt{1+4x^2}\,dx \approx 4.647.$$

Note »

When relying on technology, it is a good idea to check whether an answer is plausible. In Example 3, we found the arc length of $y = x^2$ on $[0, 2]$ is approximately 4.647. The straight-line distance between $(0, 0)$ and $(2, 4)$ is $\sqrt{20} \approx 4.472$, so our answer is reasonable.

Related Exercises 21, 24 ◆

Arc Length for $x = g(y)$ »

Sometimes it is advantageous to describe a curve as a function of y—that is, $x = g(y)$. The arc length formula in this case is derived exactly as in the case of $y = f(x)$, switching the roles of x and y. The result is the following arc length formula.

DEFINITION **Arc Length for $x = g(y)$**

Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1+g'(y)^2}\,dy.$$

Quick Check 2 What does the arc length formula give for the length of the line $x = y$ between $y = c$ and $y = d$, where $d \geq c$? Is the result consistent with the result given by the Pythagorean theorem? ◆

Answer »

$\sqrt{2}\,(d - c)$ (the length of the line segment joining the points)

EXAMPLE 4 **Arc length**

Find the length of the curve $y = f(x) = x^{2/3}$ between $x = 0$ and $x = 8$ (Figure 6.58).

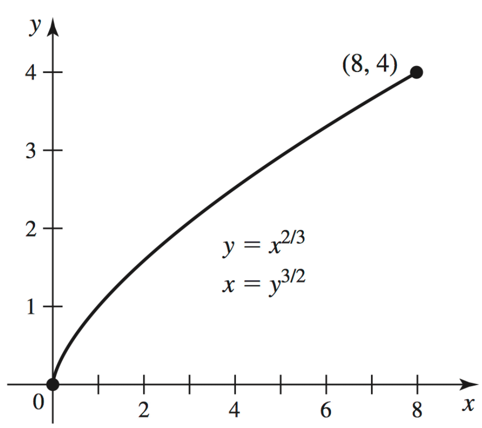


Figure 6.58

SOLUTION »

The derivative of $f(x) = x^{2/3}$ is $f'(x) = \frac{2}{3}x^{-1/3}$, which is undefined at $x = 0$. Therefore, the arc length formula with respect to x cannot be used, yet the curve certainly appears to have a well-defined length.

The key is to describe the curve with y as the independent variable. Solving $y = x^{2/3}$ for x, we have $x = g(y) = \pm y^{3/2}$. Notice that when $x = 8$, $y = 8^{2/3} = 4$, which says that we should use the positive branch of $\pm y^{3/2}$. Therefore, finding the length of the curve $y = f(x) = x^{2/3}$ from $x = 0$ to $x = 8$ is equivalent to finding the length of the curve $x = g(y) = y^{3/2}$ from $y = 0$ to $y = 4$. This is precisely the problem solved in Example 1. The arc length is $\frac{8}{27}\left(10^{3/2} - 1\right) \approx 9.1$ units.

Related Exercise 18–19 ◆

Quick Check 3 Write the integral for the length of the curve $x = \sin y$ on the interval $0 \le y \le \pi$. ◆

Answer »

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 y}\, dy$$

Exercises »

Getting Started »

Practice Exercises »

9–20. Arc length calculations *Find the arc length of the following curves on the given interval.*

9. $y = -8x - 3$ on $[-2, 6]$ (Use calculus.)

10. $y = \frac{x^3}{3} + \frac{1}{4x}$ on $[1, 5]$

11. $y = \frac{1}{3}x^{3/2}$ on $[0, 60]$

12. $y = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3}$ on $[1, 3]$

13. $y = \frac{\left(x^2 + 2\right)^{3/2}}{3}$ on $[0, 1]$

14. $y = \frac{x^{3/2}}{3} - x^{1/2}$ on $[4, 16]$

15. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on $[1, 2]$

16. $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ on $[1, 9]$

17. $x = \dfrac{(y^2 - 2)^{3/2}}{3}$, for $3 \le y \le 6$

18. $x = \dfrac{9}{4} y^{2/3} - \dfrac{1}{8} y^{4/3}$, for $1 \le y \le 2$

19. $x = 2y - 4$, for $-3 \le y \le 4$ (Use calculus, but check your work using geometry.)

20. $x = \dfrac{y^5}{5} + \dfrac{1}{12 y^3}$, for $2 \le y \le 4$

T 21–30. Arc length by calculator

a. *Write and simplify the integral that gives the arc length of the following curves on the given interval.*

b. *If necessary, use technology to evaluate or approximate the integral.*

21. $y = x^2$, for $-1 \le x \le 1$

22. $y = \sin x$, for $0 \le x \le \pi$

23. $y = \tan x$, for $0 \le x \le \pi/4$

24. $y = \dfrac{x^3}{3}$, for $-1 \le x \le 1$

25. $x = \sqrt{y - 2}$, for $3 \le y \le 4$

26. $y = \dfrac{8}{x^2}$, for $1 \le x \le 4$

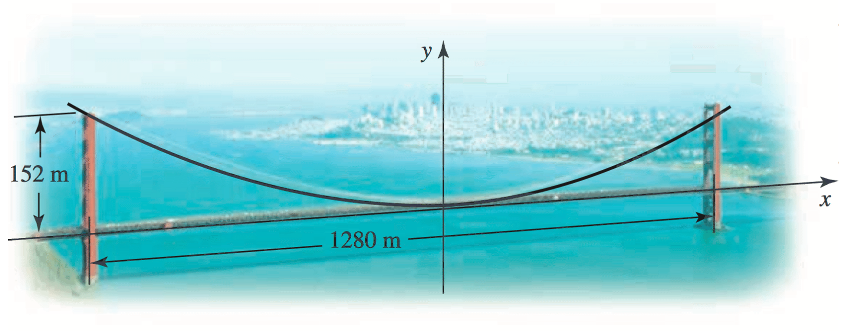
27. $y = \cos 2x$, for $0 \le x \le \pi$

28. $y = 4x - x^2$, for $0 \le x \le 4$

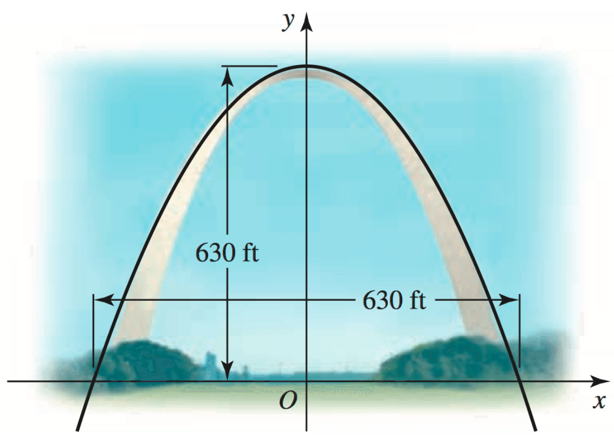
29. $y = \dfrac{1}{x}$, for $1 \le x \le 10$

30. $x = \dfrac{1}{y^2 + 1}$, for $-5 \le y \le 5$

T 31. Golden Gate cables The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge (see figure) is 1280 m long and 152 m high. The parabola $y = 0.00037 x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.



T 32. Gateway Arch The shape of the Gateway Arch in St. Louis (with a height and a base length of 630 ft) is modeled by the function $y = 630\left(1 - \left(\frac{x}{315}\right)^2\right)$, where $|x| \le 315$, and x and y are measured in feet (see figure). Estimate the length of the Gateway Arch.



33. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_a^b (1 + f'(x))\, dx$

b. Assuming f' is continuous on the interval $[a, b]$, the length of the curve $y = f(x)$ on $[a, b]$ is the area under the curve $y = \sqrt{1 + f'(x)^2}$ on $[a, b]$.

c. Arc length may be negative if $f(x) < 0$ on part of the interval in question.

34. Arc length for a line Consider the segment of the line $y = mx + c$ on the interval $[a, b]$. Use the arc length formula to show that the length of the line segment is $(b - a)\sqrt{1 + m^2}$. Verify this result by computing the length of the line segment using the distance formula.

35. Functions from arc length What differentiable functions have an arc length on the interval $[a, b]$ given by the following integrals? Note that the answers are not unique. Give a family of functions that satisfy the conditions.

a. $\int_a^b \sqrt{1 + 16x^4}\, dx$

b. $\int_a^b \sqrt{1 + 36\cos^2 2x}\ dx$

36. **Function from arc length** Find a curve passes through the point (1, 5) and has an arc length on the interval [2, 6] given by $\int_2^6 \sqrt{1 + 16\,x^{-6}}\ dx$.

T **37.** **Cosine vs. parabola** Which curve has the greater length on the interval $[-1, 1]$, $y = 1 - x^2$ or $y = \cos \frac{\pi x}{2}$?

38. **Function defined as an integral** Write the integral that gives the length of the curve $y = f(x) = \int_0^x \sin t\ dt$ on the interval $[0, \pi]$.

Explorations and Challenges »

39. **Lengths of related curves** Suppose the graph of f on the interval $[a, b]$ has length L, where f' is continuous on $[a, b]$. Evaluate the following integrals in terms of L.

a. $\int_{a/2}^{b/2} \sqrt{1 + f'(2x)^2}\ dx$

b. $\int_{a/c}^{b/c} \sqrt{1 + f'(cx)^2}\ dx$ if $c \neq 0$

40. **Lengths of symmetric curves** Suppose a curve is described by $y = f(x)$ on the interval $[-b, b]$, where f' is continuous on $[-b, b]$. Show that if f is odd *or* f is even, then the length of the curve $y = f(x)$ from $x = -b$ to $x = b$ is twice the length of the curve from $x = 0$ to $x = b$. Use a geometric argument and prove it using integration.

41. **A family of algebraic functions**

a. Show that the arc length integral for the function $f(x) = a\,x^n + \frac{1}{4\,a\,n\,(n-2)\,x^{n-2}}$, where a and n are positive real numbers with $n \neq 2$, may be integrated using methods you already know.

b. Verify that the arc length of the curve $y = f(x)$ on the interval [1, 2] is

$$a\,(2^n - 1) + \frac{1 - 2^{2-n}}{4\,a\,n\,(n-2)}.$$

T **42.** **Bernoulli's "parabolas"** Johann Bernoulli (1667–1748) evaluated the arc length of curves of the form $y = x^{(2n+1)/2n}$, where n is a positive integer, on the interval $[0, a]$.

a. Write the arc length integral.

b. Make the change of variables $u^2 = 1 + \left(\frac{2n+1}{2n}\right)^2 x^{1/n}$ to obtain a new integral with respect to u.

c. Use the Binomial Theorem to expand this integrand and evaluate the integral.

d. The case $n = 1$ ($y = x^{3/2}$) was done in Example 1. With $a = 1$, compute the arc length in the cases $n = 2$ and $n = 3$. Does the arc length increase or decrease with n?

e. Graph the arc length of the curves for $a = 1$ as a function of n (treat n as a real number when graphing).