6.4 Volume by Shells

You can solve many challenging volume problems using the disk/washer method. There are, however, some volume problems that are difficult to solve with this method. For this reason, we extend our discussion of volume problems to the *shell method*, which—like the disk/washer method—is used to compute the volume of solids of revolution.

Note »

Why another method? Suppose R is the region in the first quadrant bounded by the graph of $y = x^2 - x^3$ and the x-axis (**Figure 6.38**). When R is revolved about the y-axis, the resulting solid has a volume that is difficult to compute using the washer method. The volume is much easier to compute using the shell method.

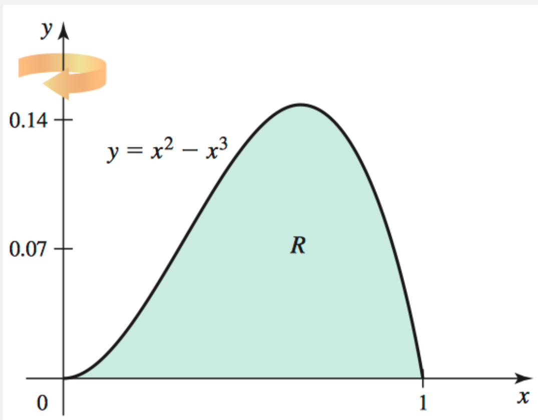


Figure 6.38

Cylindrical Shells »

Let R be a region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$, where $f(x) \geq 0$ on $[a, b]$. When R is revolved about the y-axis, a solid is generated (**Figure 6.39**) whose volume is computed with the slice-and-sum strategy.

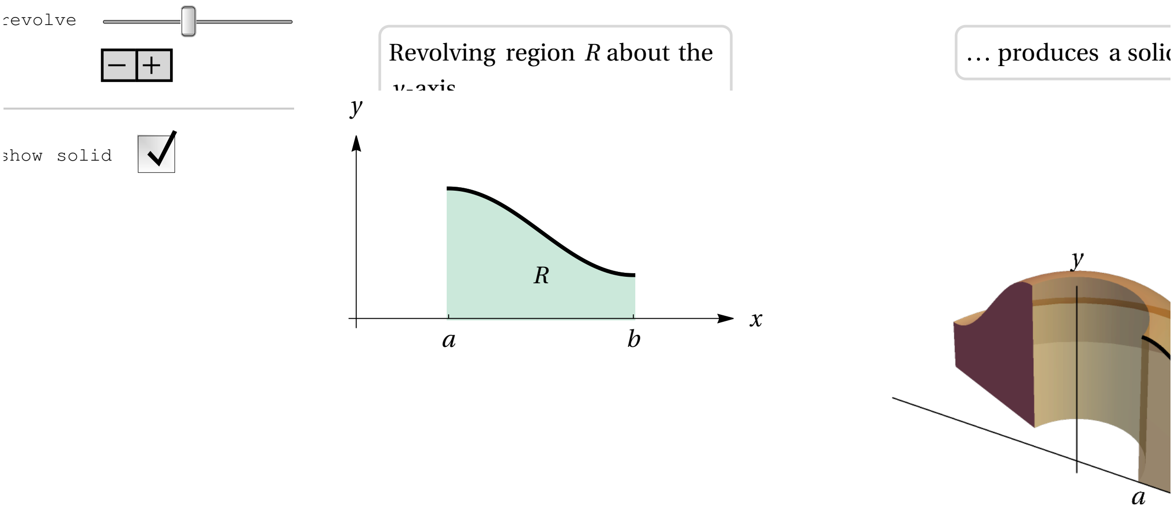


Figure 6.39

We divide $[a, b]$ into n subintervals of length $\Delta x = \dfrac{b-a}{n}$, and identify an arbitrary point x_k^* on the kth subinterval for $k = 1, \ldots, n$. Now observe the rectangle built on the kth subinterval with a height of $f(x_k^*)$ and a width Δx (**Figure 6.40**). As it revolves about the y-axis, this rectangle sweeps out a thin *cylindrical shell*.

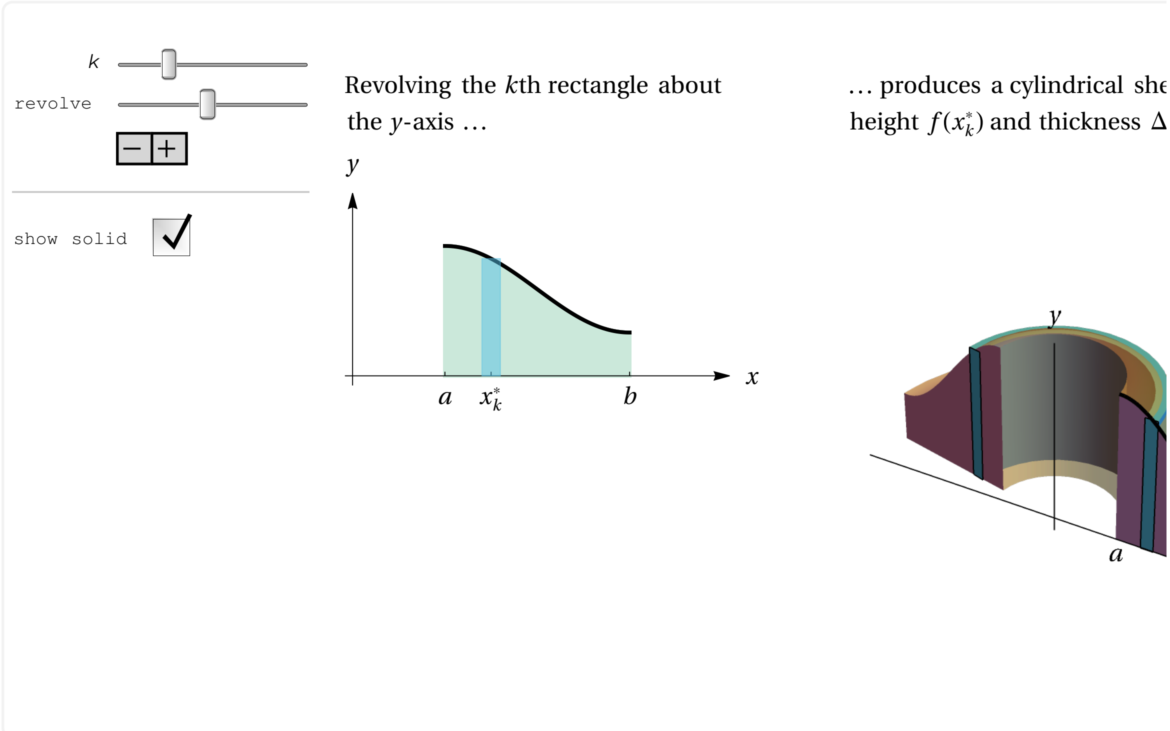


Figure 6.40

When the kth cylindrical shell is unwrapped (**Figure 6.41**), it approximates a thin rectangular slab. The approximate length of the slab is the circumference of a circle with radius x_k^*, which is $2\pi x_k^*$. The height of the slab is the height of the original rectangle $f(x_k^*)$ and its thickness is Δx; therefore, the volume of the kth shell is approximately

$$\underbrace{2\pi x_k^*}_{\text{length}} \cdot \underbrace{f(x_k^*)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{thickness}} = 2\pi x_k^* \, f(x_k^*)\,\Delta x.$$

Summing the volumes of the n cylindrical shells gives an approximation to the volume of the entire solid:

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* \, f(x_k^*)\,\Delta x.$$

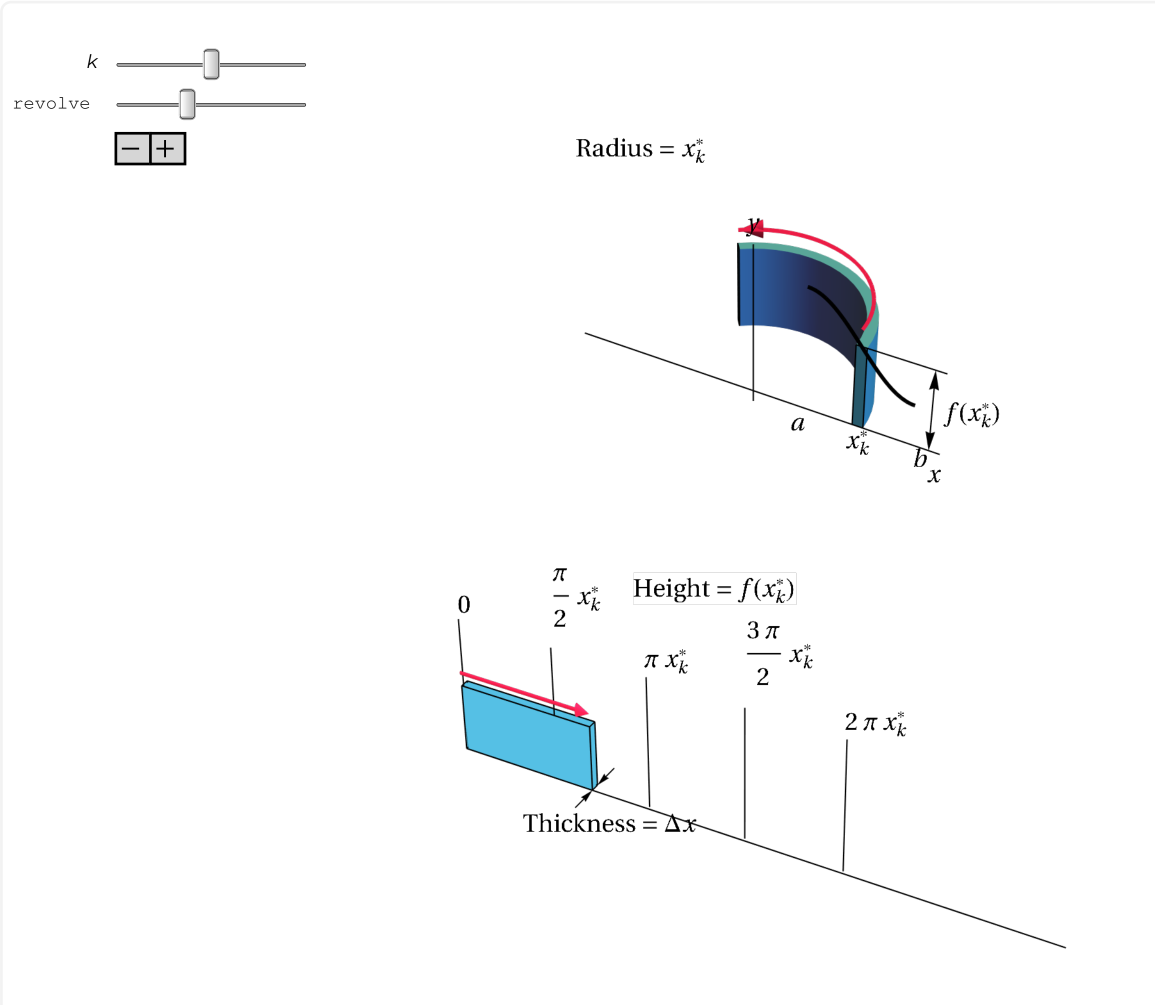


Figure 6.41

As n increases and as Δx approaches 0 (**Figure 6.42**), we obtain the exact volume of the solid as a definite integral:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{2\,\pi\, x_k^*}_{\substack{\text{shell}\\ \text{circumference}}} \overbrace{f(x_k^*)}^{\substack{\text{shell}\\ \text{height}}} \underbrace{\Delta x}_{\substack{\text{shell}\\ \text{thickness}}} = \int_a^b 2\,\pi\, x\, f(x)\, dx.$$

Note »

Rather than memorizing, think of the meaning of the factors in this formula: $f(x)$ is the height of a single cylindrical shell, $2\,\pi\, x$ is the circumference of the shell, and dx corresponds to the thickness of a shell. Therefore, $2\,\pi\, x\, f(x)\, dx$ represents the volume of a single shell, and we sum the volumes from $x = a$ to $x = b$. Notice that the integrand for the shell method is the function $A(x)$ that gives the surface area of the shell of radius x, for $a \le x \le b$.

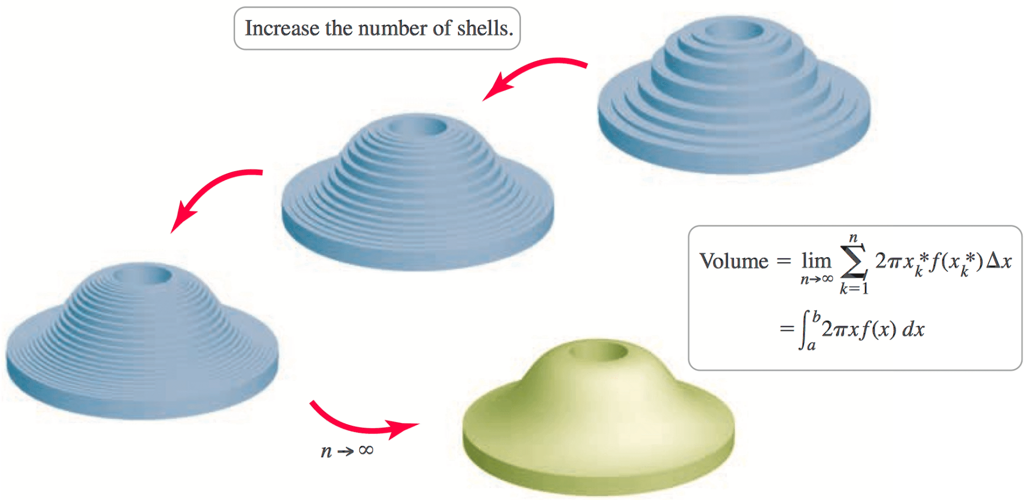


Figure 6.42

Before doing examples, we generalize this method as we did for the disk method. Suppose the region R is bounded by two curves, $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ on $[a, b]$ (**Figure 6.43**). What is the volume of the solid generated when R is revolved about the y-axis?

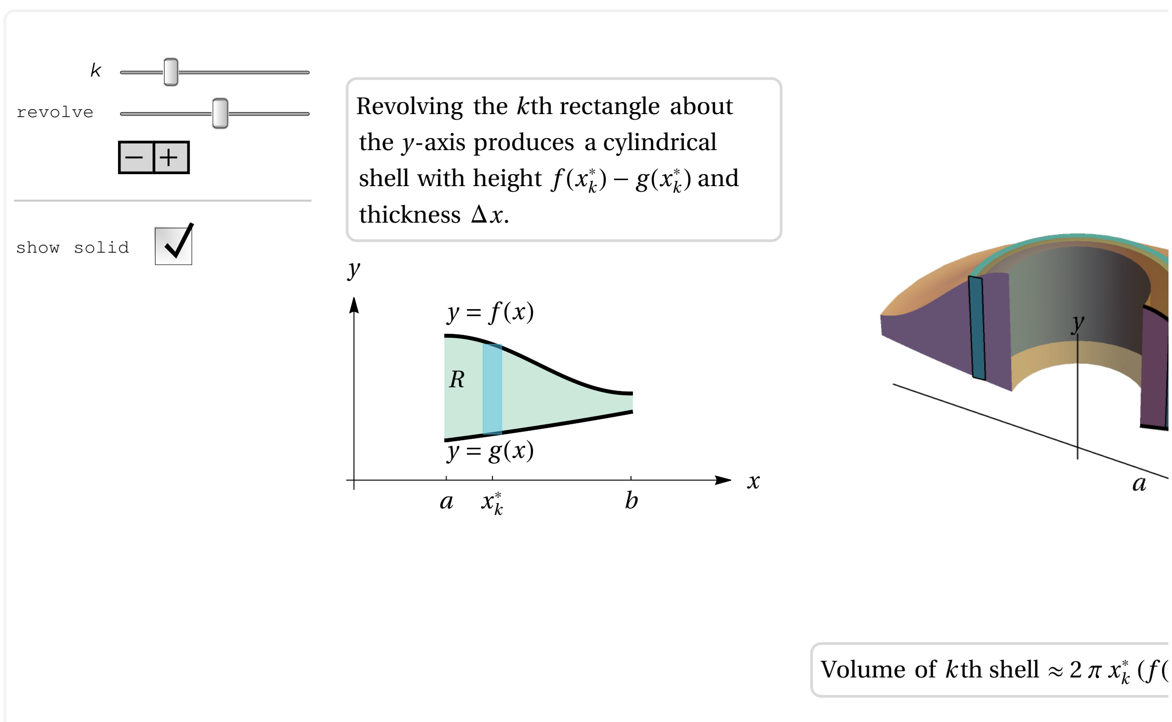


Figure 6.43

The situation is similar to the case we just considered. A typical rectangle in R sweeps out a cylindrical shell, but now the height of the kth shell is $f(x_k^*) - g(x_k^*)$, for $k = 1, \ldots, n$. As before, we take the radius of the kth shell to be x_k^*, which means the volume of the kth shell is approximated by $2\pi x_k^* (f(x_k^*) - g(x_k^*)) \Delta x$ (Figure 6.43). Summing the volumes of all the shells gives an approximation to the volume of the entire solid:

$$V \approx \sum_{k=1}^{n} \underbrace{2\,\pi\,x_k^*}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(f(x_k^*) - g(x_k^*))}_{\substack{\text{shell}\\ \text{height}}}\,\Delta x.$$

Taking the limit as $n \to \infty$ (which implies that $\Delta x \to 0$), the volume is the definite integral

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\,\pi\,x_k^*\,(f(x_k^*) - g(x_k^*))\,\Delta x = \int_a^b 2\,\pi\,x\,(f(x) - g(x))\,dx.$$

We now have the formula for the shell method.

Volume by the Shell Method

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a,\ b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b \underbrace{2\,\pi\,x}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(f(x) - g(x))}_{\substack{\text{shell}\\ \text{height}}}\,dx.$$

Note »

An analogous formula for the shell method when R is revolved about the x-axis is obtained by reversing the roles of x and y:

$$V = \int_c^d 2\,\pi\,y\,(p(y) - q(y))\,dy.$$

We assume R is bounded by the curves $x = p(y)$ and $x = q(y)$, where $p(y) \geq q(y)$ on $[c,\ d]$.

EXAMPLE 1 A sine bowl

Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$ (**Figure 6.44**). Find the volume of the solid generated when R is revolved about the y-axis.

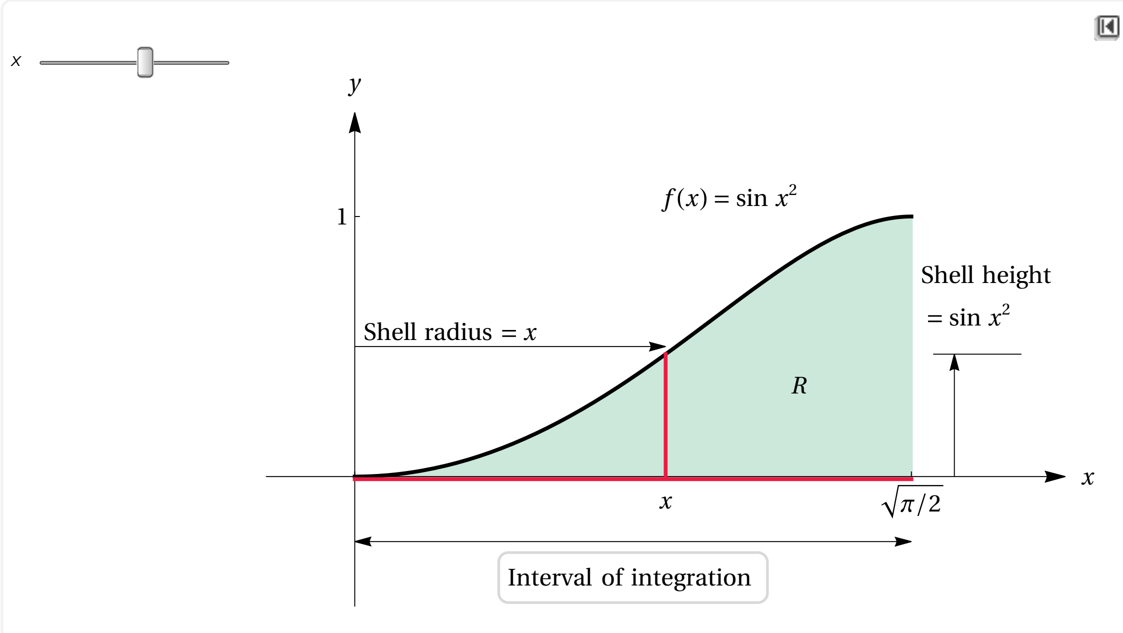


Figure 6.44

Note »

> When computing volumes using the shell method, it is best to sketch the region R in the xy-plane and draw a slice through the region that generates a typical shell.

SOLUTION »

Revolving R about the y-axis produces a bowl-shaped region (**Figure 6.45**). The radius of a typical cylindrical shell is x and its height is $f(x) = \sin x^2$. Therefore, the volume by the shell method is

$$V = \int_a^b \underbrace{2\pi x}_{\substack{\text{shell}\\\text{circumference}}} \underbrace{f(x)}_{\substack{\text{shell}\\\text{height}}} \, dx = \int_0^{\sqrt{\pi/2}} 2\pi x \sin x^2 \, dx.$$

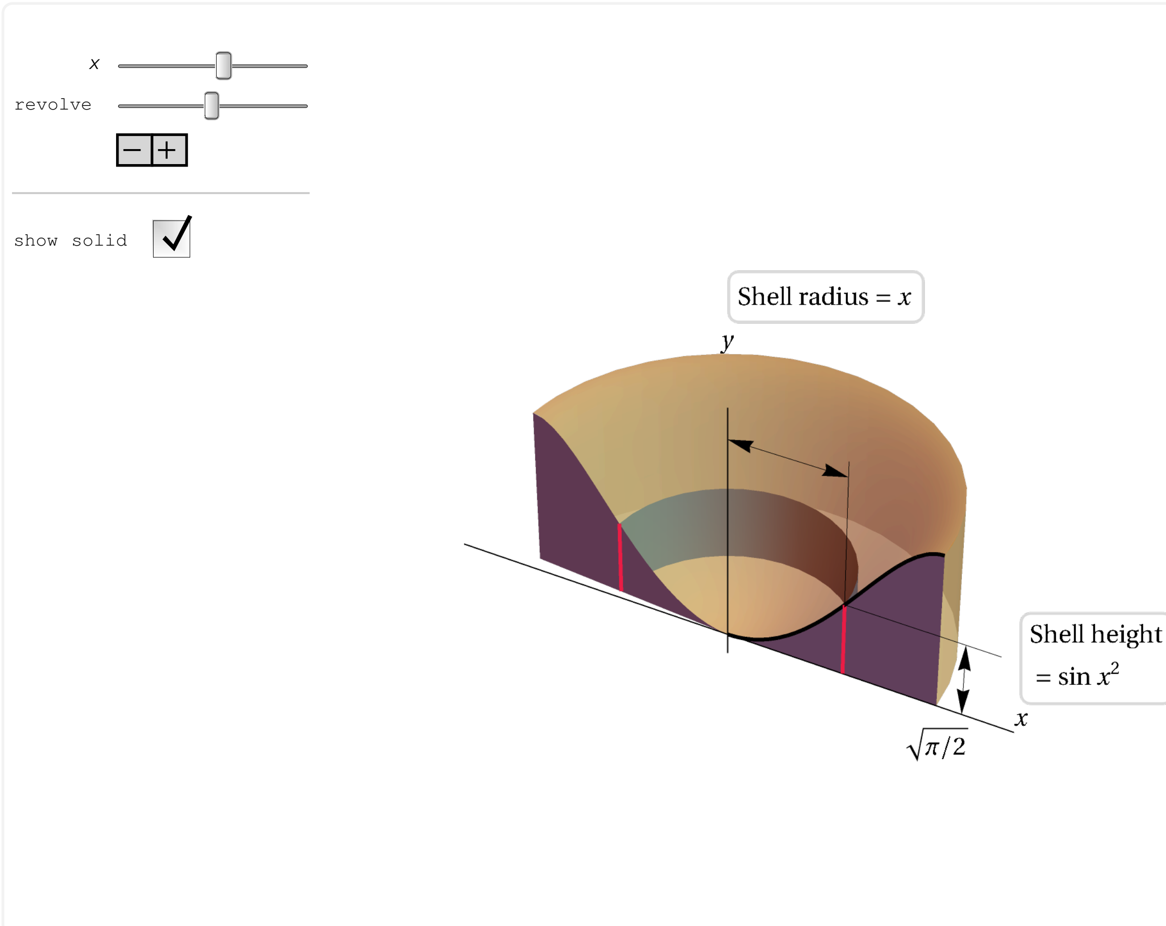


Figure 6.45

Now we make the change of variables $u = x^2$, which means that $du = 2x\,dx$. The lower limit $x = 0$ becomes $u = 0$ and the upper limit $x = \sqrt{\pi/2}$ becomes $u = \dfrac{\pi}{2}$. The volume of the solid is

$$V = \int_0^{\sqrt{\pi/2}} \underbrace{2\pi x}_{\substack{\text{shell}\\\text{circumference}}} \underbrace{\sin x^2}_{\substack{\text{shell}\\\text{height}}}\,dx = \pi \int_0^{\pi/2} \sin u\,du \quad u = x^2,\ du = 2x\,dx$$

$$= \pi(-\cos u)\Big|_0^{\pi/2} \quad \text{Fundamental Theorem}$$

$$= \pi[0 - (-1)] = \pi. \quad \text{Simplify.}$$

Related Exercises 10, 17 ◆

Quick Check 1 The triangle bounded by the x-axis, the line $y = 2x$, and the line $x = 1$ is revolved about the y-axis. Give an integral that equals the volume of the resulting solid using the shell method? ◆

Answer »

$$\int_0^1 2\pi x(2x)\,dx$$

EXAMPLE 2 Shells about the x-axis

Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and the line $y = 2$. Find the volume of the solid generated when R is revolved about the x-axis.

Note »

> In Example 2, we could use the disk/washer method to compute the volume, but notice that this approach requires splitting the region into two subregions. A better approach is to use the shell method and integrate along the y-axis.

SOLUTION »

The revolution is about the x-axis, so the integration in the shell method is with respect to y. A typical shell runs parallel to the x-axis and has radius y, where $0 \le y \le 2$; the shells extend from the y-axis to the curve $y = \sqrt{x-2}$ (**Figure 6.46**).

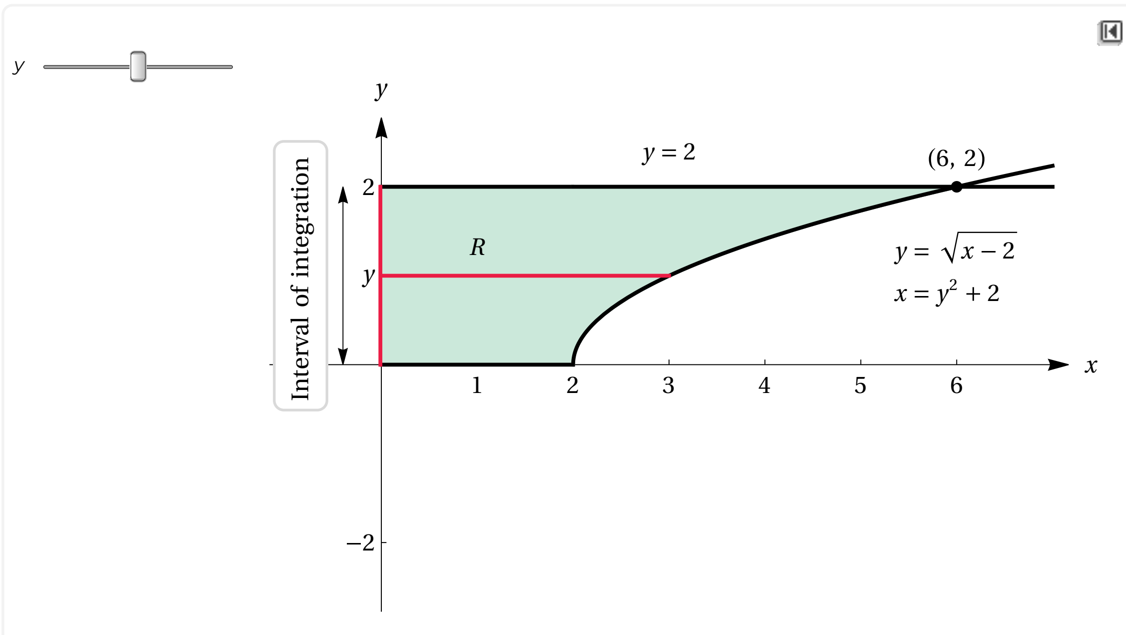


Figure 6.46

Solving $y = \sqrt{x-2}$ for x, we have $x = y^2 + 2$, which is the height of the shell at the point y (**Figure 6.47**). Integrating with respect to y, the volume of the solid is

$$V = \int_0^2 \underbrace{2\pi y}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{\left(y^2+2\right)}_{\substack{\text{shell}\\ \text{height}}} dy = 2\pi \int_0^2 \left(y^3 + 2y\right) dy = 16\pi.$$

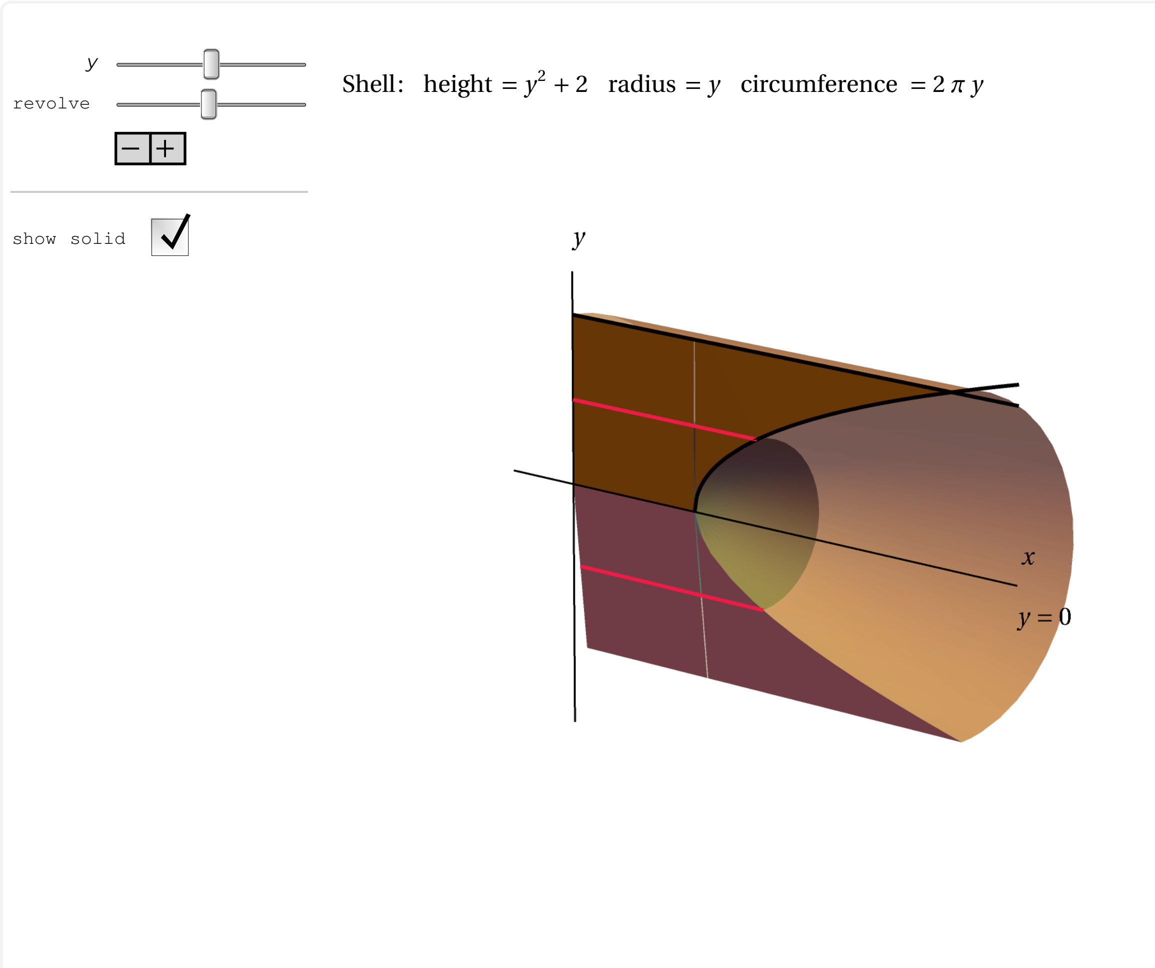


Figure 6.47

Related Exercises 13, 21 ◆

EXAMPLE 3 Volume of a drilled sphere

A cylindrical hole with radius r is drilled symmetrically through the center of a sphere with radius a, where $0 \le r \le a$. What is the volume of the remaining material?

SOLUTION »

The y-axis is chosen to coincide with the axis of the cylindrical hole. We let R be the region in the xy-plane bounded above by $f(x) = \sqrt{a^2 - x^2}$, the upper half of a circle of radius a, and bounded below by $g(x) = -\sqrt{a^2 - x^2}$, the lower half of a circle of radius a, for $r \le x \le a$ (**Figure 6.48**).

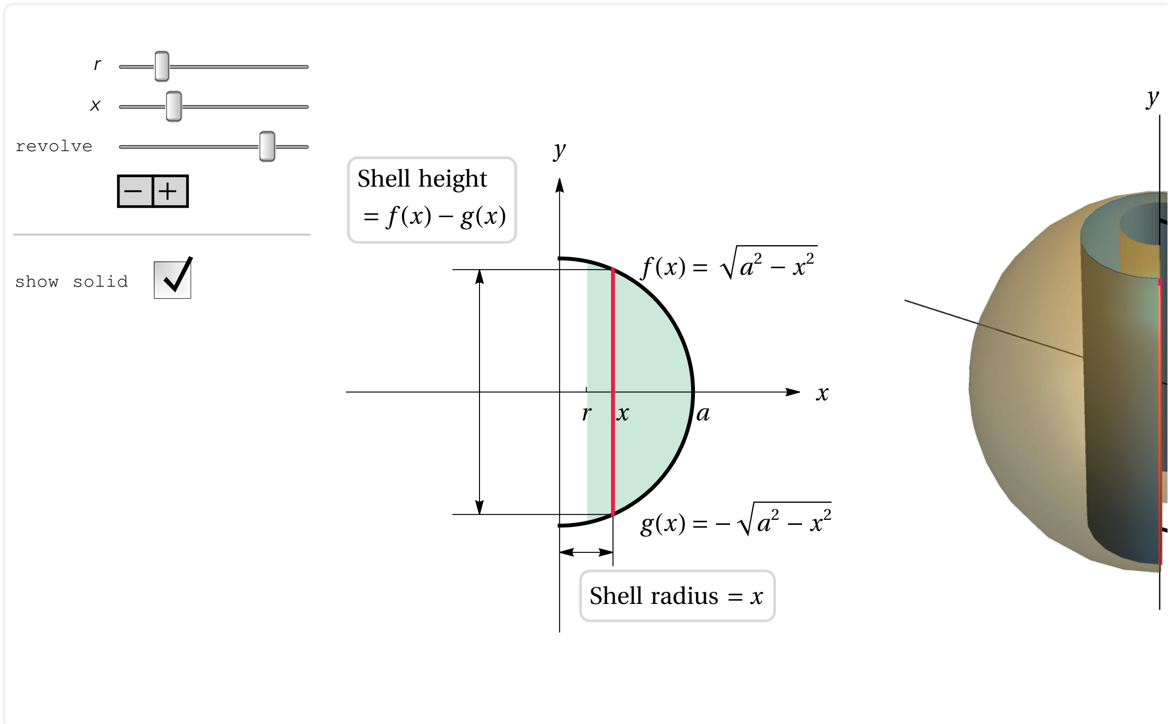


Figure 6.48

Slices are taken perpendicular to the x-axis from $x = r$ to $x = a$. When a slice is revolved about the y-axis, it sweeps out a cylindrical shell that is concentric with the hole through the sphere. The radius of a typical shell is x and its height is $f(x) - g(x) = 2\sqrt{a^2 - x^2}$. Therefore, the volume of the material that remains in the sphere is

$$
\begin{aligned}
V &= \int_r^a \underbrace{2\pi x}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{\left(2\sqrt{a^2 - x^2}\right)}_{\substack{\text{shell}\\ \text{height}}} dx \\
&= -2\pi \int_{a^2 - r^2}^{0} \sqrt{u}\, du && u = a^2 - x^2,\ du = -2x\,dx \\
&= 2\pi \left(\frac{2}{3} u^{3/2}\right)\Bigg|_0^{a^2 - r^2} && \text{Fundamental Theorem} \\
&= \frac{4\pi}{3}\left(a^2 - r^2\right)^{3/2}. && \text{Simplify.}
\end{aligned}
$$

It is important to check the result by examining special cases. In the case that $r = a$ (the radius of the hole equals the radius of the sphere), our calculation gives a volume of 0, which is correct. In the case that $r = 0$ (no hole in the sphere), our calculation gives the correct volume of a sphere, $\frac{4}{3}\pi a^3$.

Related Exercises 62–63 ◆

EXAMPLE 4 Revolving about other lines

Let R be the region bounded by the curve $y = \sqrt{x}$, the line $y = 1$, and the y-axis (**Figure 6.49a**).

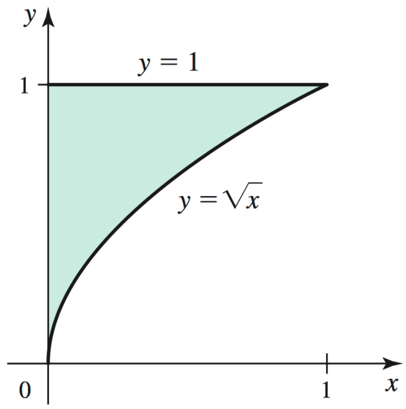


Figure 6.49 a

a. Use the shell method to find the volume of the solid generated when R is revolved about the line $x = -\frac{1}{2}$ (**Figure 6.49b**).

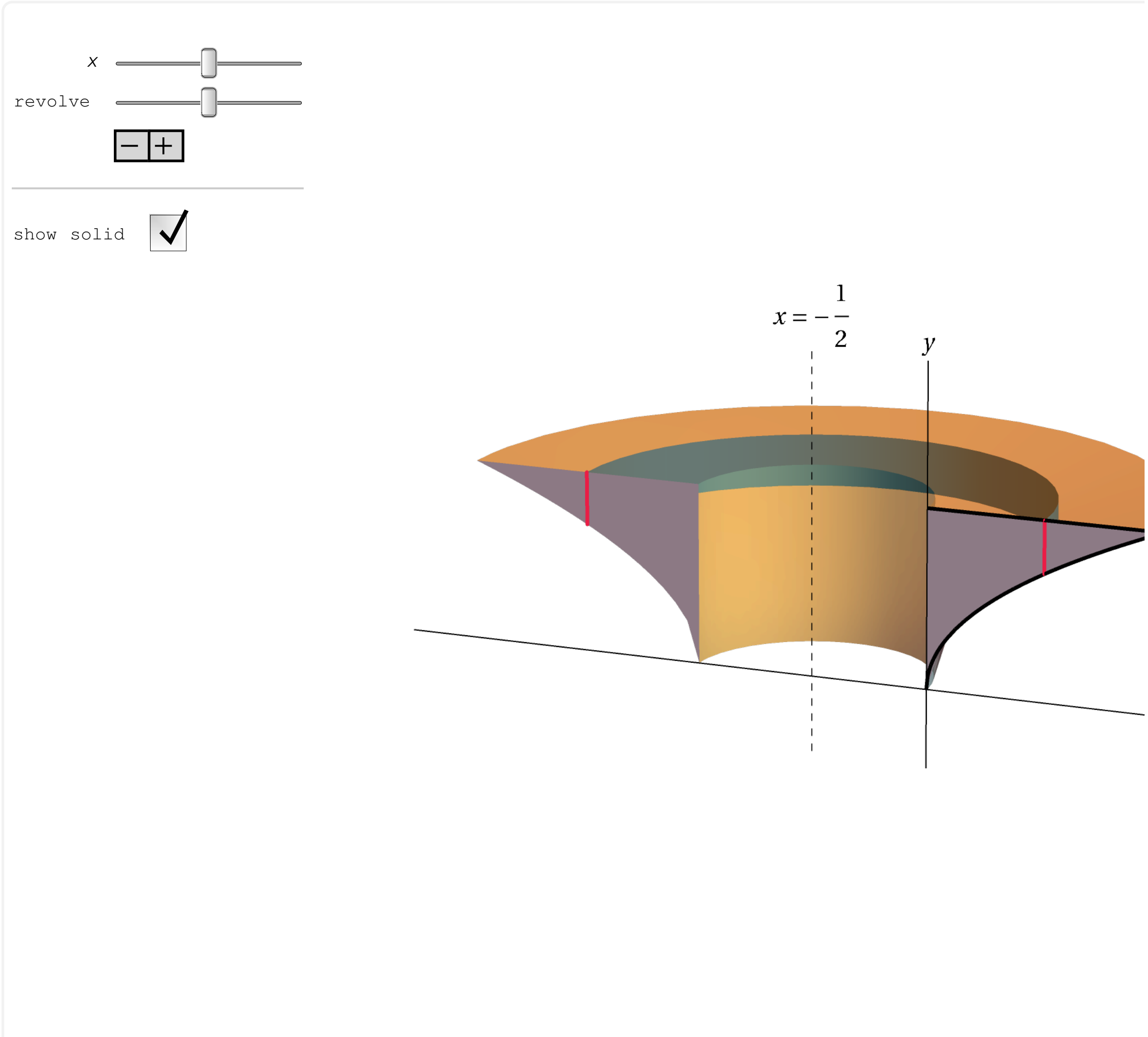


Figure 6.49 b

b. Use the disk/washer method to find the volume of the solid generated when R is revolved about the line $y = 1$ (**Figure 6.49c**).

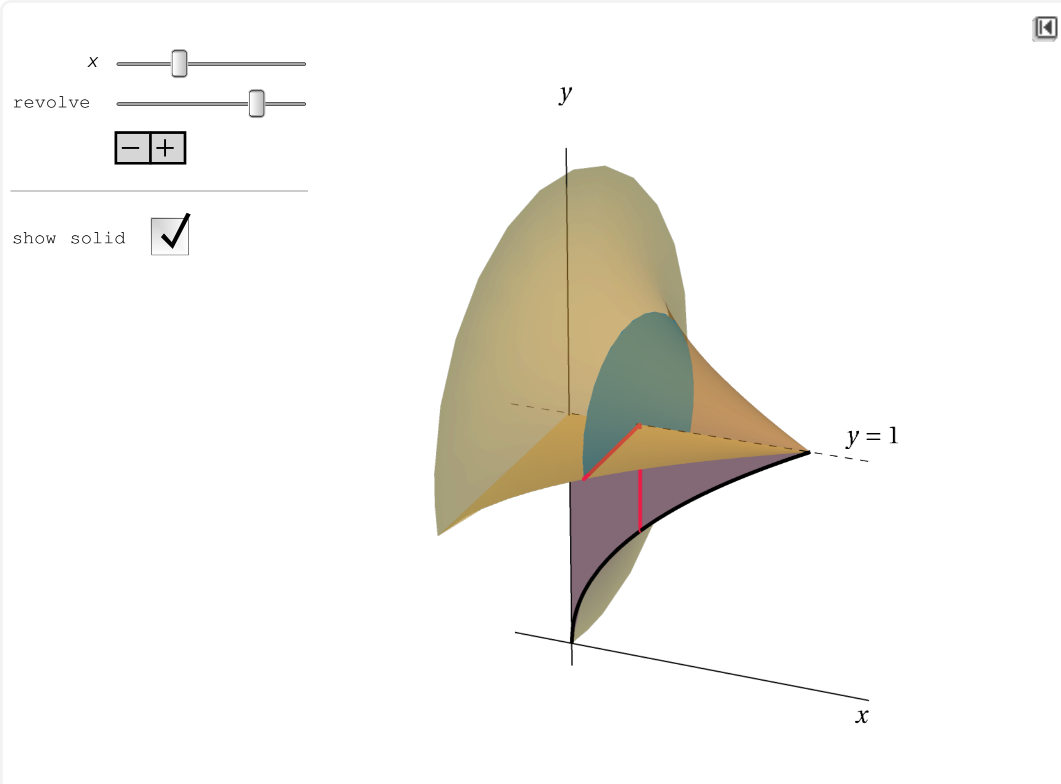


Figure 6.49 c

SOLUTION »

a. Using the shell method, we must imagine taking slices through R parallel to the y-axis. A typical slice through R at a point x, where $0 \le x \le 1$, has height $1-\sqrt{x}$. When that slice is revolved about the line $x=-\frac{1}{2}$, it sweeps out a cylindrical shell with a radius of $x+\frac{1}{2}$ and a height of $1-\sqrt{x}$ (**Figure 6.50**). A slight modification of the standard shell method gives the volume of the solid:

$$\int_0^1 \underbrace{2\pi\left(x+\frac{1}{2}\right)}_{\text{shell circumference}} \underbrace{\left(1-\sqrt{x}\right)}_{\text{height of shell}} dx = 2\pi\int_0^1 \left(x - x^{3/2} + \frac{1}{2} - \frac{x^{1/2}}{2}\right) dx \qquad \text{Expand integrand.}$$

$$= 2\pi\left(\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{2}x - \frac{1}{3}x^{3/2}\right)\bigg|_0^1 \qquad \text{Evaluate integral.}$$

$$= \frac{8\pi}{15}.$$

Note »

If we instead revolved about the y-axis ($x = 0$), the radius of the shell would be x. Because we are revolving about the line $x = -\frac{1}{2}$, the radius of the shell is $x + \frac{1}{2}$.

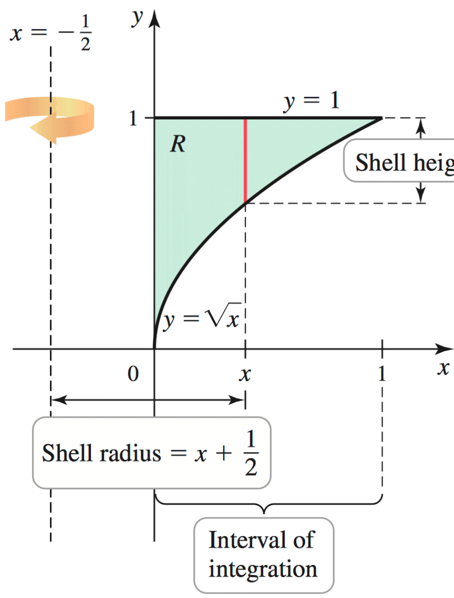


Figure 6.50

b. Using the disk/washer method, we take slices through R parallel to the y-axis. Consider a typical slice at a point x, where $0 \le x \le 1$. Its length, now measured with respect to the line $y = 1$, is $1 - \sqrt{x}$. When that slice is revolved about the line $y = 1$, it sweeps out a disk of radius $1 - \sqrt{x}$ (**Figure 6.51**). By the disk/washer formula, the volume of the solid is

$$\int_0^1 \pi \underbrace{\left(1 - \sqrt{x}\right)^2}_{\text{radius of disk}} dx = \pi \int_0^1 \left(1 - 2\sqrt{x} + x\right) dx \quad \text{Expand integrand.}$$

$$= \pi \left(x - \frac{4}{3} x^{3/2} + \frac{1}{2} x^2 \right) \Big|_0^1 \quad \text{Evaluate integral.}$$

$$= \frac{\pi}{6}.$$

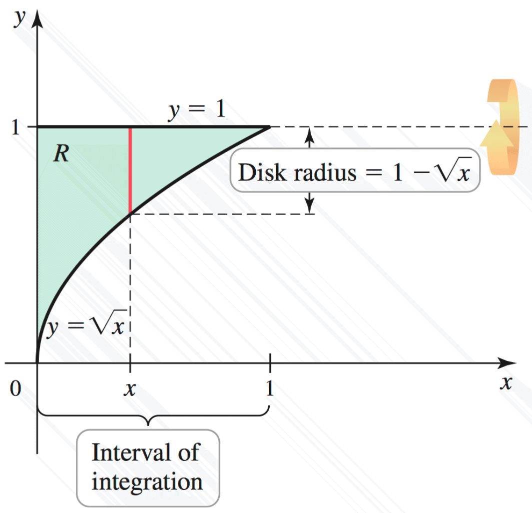


Figure 6.51

Note »

> The disk/washer method can also be used for part (a) and the shell method can also be used for part (b).

Related Exercises 32–33 ◆

Quick Check 2 Write the volume integral in Example 4b in the case that R is revolved about the line $y = -5$. ◆

Answer »

$$V = \int_0^1 \pi\left(36 - \left(\sqrt{x} + 5\right)^2\right) dx$$

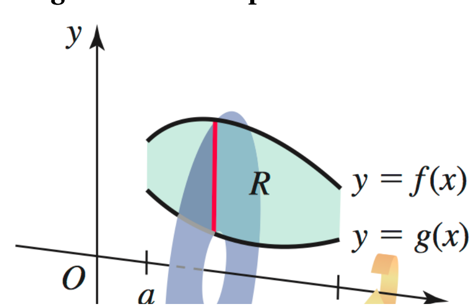
Restoring Order »

After working with slices, disks, washers, and shells, you may feel somewhat overwhelmed. How do you choose a method and which method is best?

First, notice that the disk method is just a special case of the washer method. So, for solids of revolution, the choice is between the washer method and the shell method. In *principle*, either method can be used. In *practice*, one method usually produces an integral that is easier to evaluate than the other method. The following table summarizes these methods.

SUMMARY **Disk/Washer and Shell Methods**

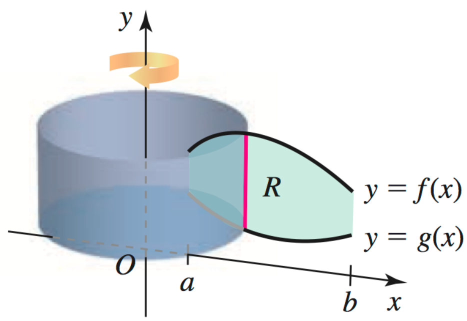
Integration with respect to x



Disk/washer method about the x–axis

Disks/washers are *perpendicular* to the x–axis.

$$\int_a^b \pi\,\underbrace{f(x)}_{\text{outer radius}}{}^2 - \pi\,\underbrace{g(x)}_{\text{inner radius}}{}^2\,dx$$

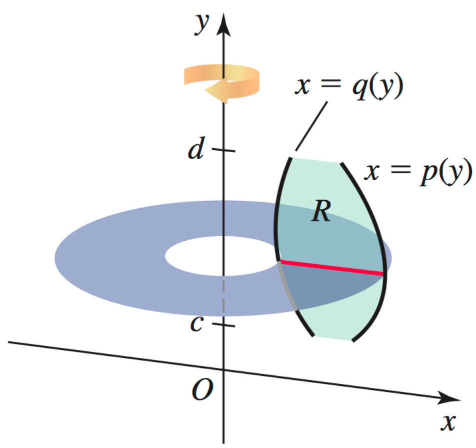


Shell method about the *y*–axis

Shells are *parallel* to the *y*–axis.

$$\int_a^b \underbrace{2\pi x}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(f(x)-g(x))}_{\substack{\text{shell}\\ \text{height}}}\,dx$$

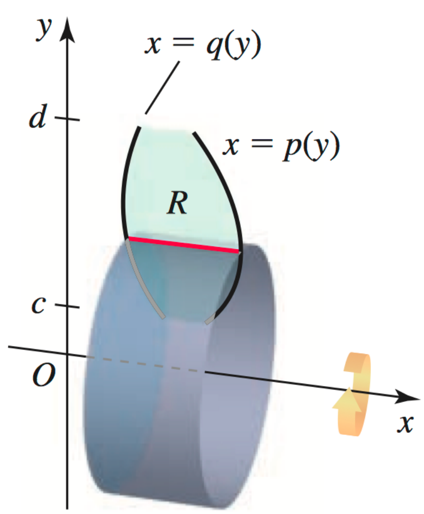
Integration with respect to *y*



Disk/washer method about the *y*–axis

Disks/washers are *perpendicular* to the *y*–axis.

$$\int_c^d \pi\ \underbrace{p(y)}_{\substack{\text{outer}\\ \text{radius}}}{}^2 - \pi\ \underbrace{q(y)}_{\substack{\text{inner}\\ \text{radius}}}{}^2\ dy$$



Shell method about the *x*–axis

Shells are *parallel* to the *x*–axis.

$$\int_c^d \underbrace{2\pi y}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{(p(y)-q(y))}_{\substack{\text{shell}\\ \text{height}}}\,dy$$

The following example shows that while two methods may be used on the same problem, one of them may be preferable.

EXAMPLE 5 **Volume by which method?**

The region R is bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x$ on the interval $[0, 1]$ (**Figure 6.52**). Use the washer method and the shell method to find the volume of the solid formed when R is revolved about the x-axis.

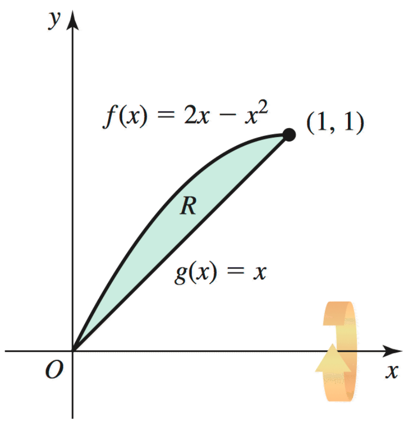


Figure 6.52

SOLUTION »

Solving $f(x) = g(x)$, we find that the curves intersect at the points $(0, 0)$ and $(1, 1)$. Using the washer method, the upper bounding curve is the graph of f, the lower bounding curve is the graph of g, and a typical washer is perpendicular to the x-axis (**Figure 6.53**). Therefore, the volume is

$$V = \int_0^1 \pi \underbrace{\left(2x - x^2\right)}_{\substack{\text{outer}\\\text{radius}}}^2 - \pi \underbrace{x}_{\substack{\text{inner}\\\text{radius}}}^2\, dx \quad \text{Washer method}$$

$$= \pi \int_0^1 \left(x^4 - 4x^3 + 3x^2\right) dx \quad \text{Expand integrand.}$$

$$= \pi \left(\frac{x^5}{5} - x^4 + x^3\right)\Bigg|_0^1 = \frac{\pi}{5}. \quad \text{Evaluate integral.}$$

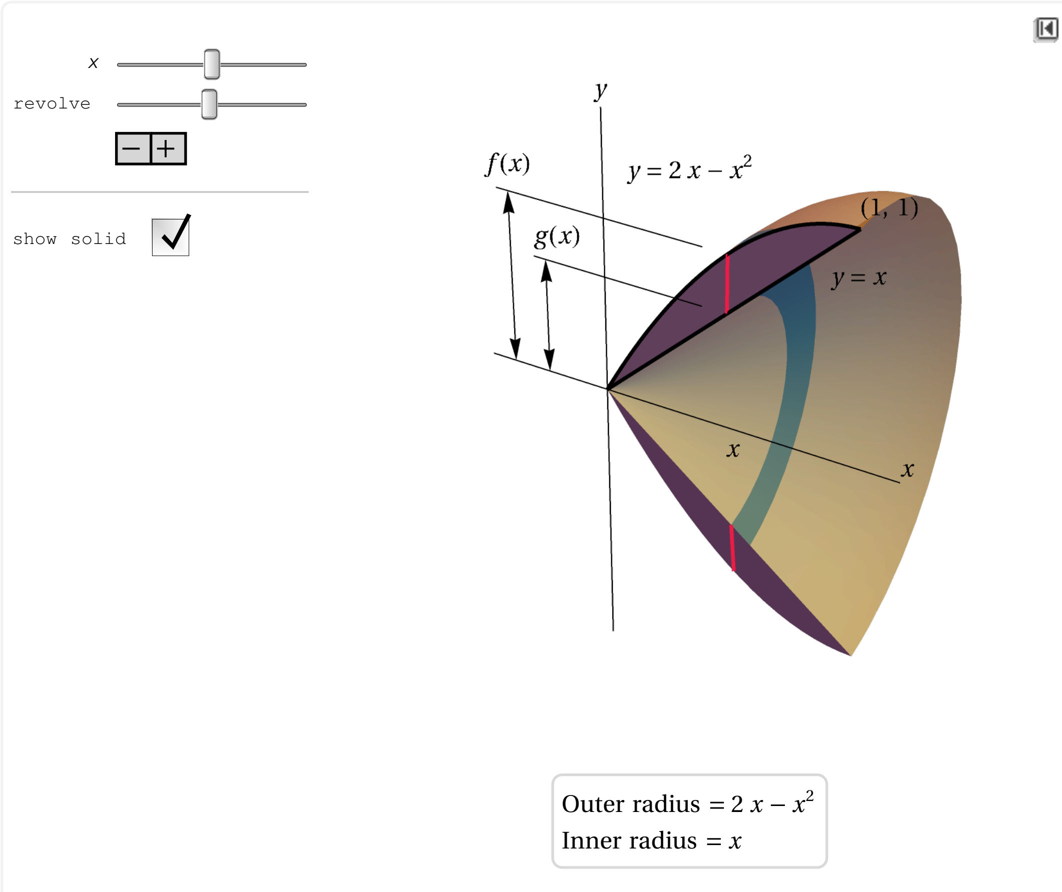


Figure 6.53

The shell method requires expressing the bounding curves in the form $x = p(y)$ for the right curve and $x = q(y)$ for the left curve. The right curve is $x = y$. Solving $y = 2x - x^2$ for x, we find that $x = 1 - \sqrt{1-y}$ describes the left curve. A typical shell is parallel to the x-axis (**Figure 6.54**). Therefore, the volume is

$$V = \int_0^1 \underbrace{2\pi y}_{\substack{\text{shell}\\ \text{circumference}}} \underbrace{\left(y - \left(1 - \sqrt{1-y}\right)\right)}_{\substack{\text{shell}\\ \text{height}}} dy.$$

This integral equals $\dfrac{\pi}{5}$, but it is more difficult to evaluate than the integral required by the washer method. In this case, the washer method is preferable. Of course, the shell method may be preferable for other problems.

Note »

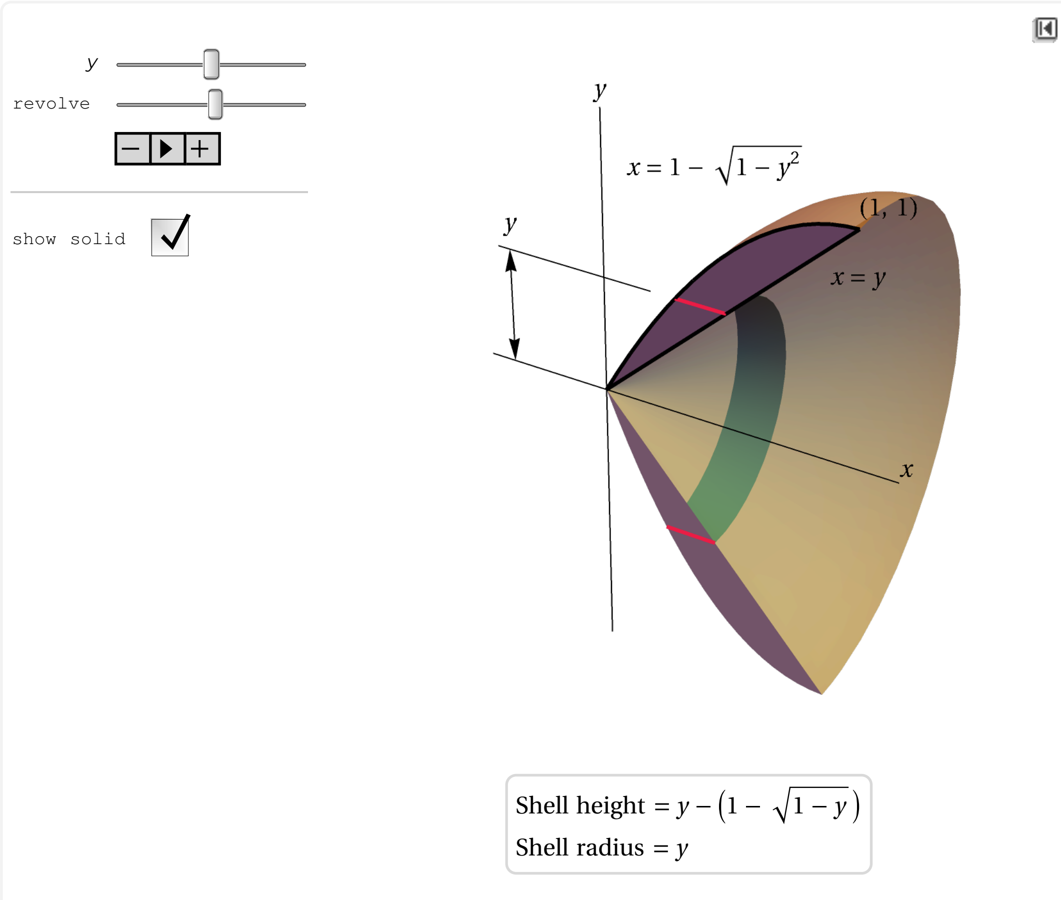


Figure 6.54

Related Exercises 49–50 ◆

Quick Check 3 Suppose the region in Example 5 is revolved about the y–axis. Which method (washer or shell) leads to an easier integral? ◆

Answer »

The shell method is easier.

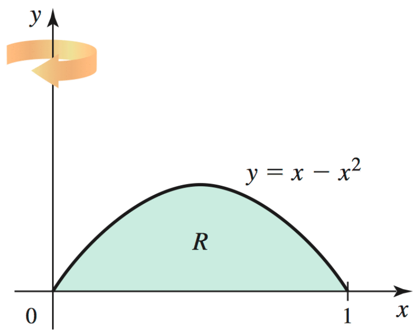
Exercises »

Getting Started »

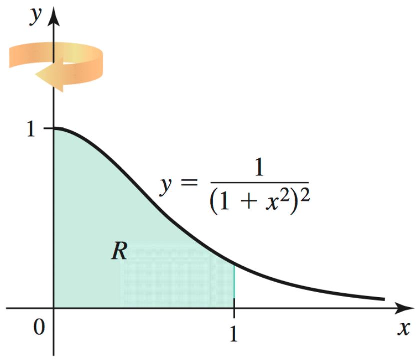
Practice Exercises »

9–30. Shell method *Let R be the region bounded by the following curves. Use the shell method to find the volume of the solid generated when R is revolved about indicated axis.*

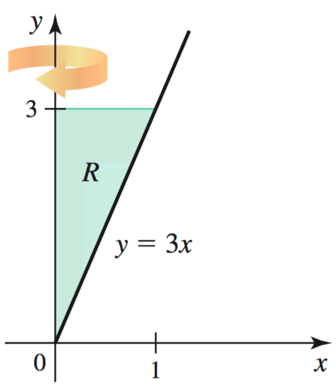
9. $y = x - x^2$ and $y = 0$; about the y-axis



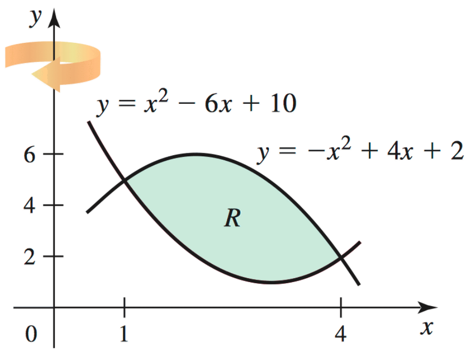
10. $y = \left(1 + x^2\right)^{-2}$, $y = 0$, $x = 0$, and $x = 1$; about the y-axis



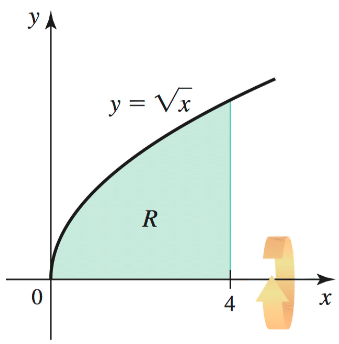
11. $y = 3x$, $y = 3$, and $x = 0$; about the y-axis (Use integration and check your answer using the volume formula for a cone.)



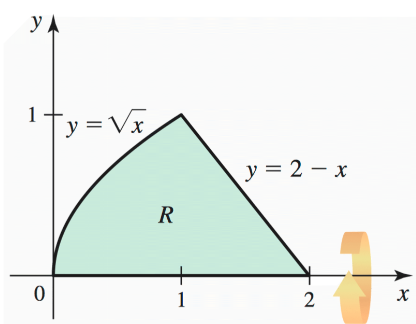
12. $y = -x^2 + 4x + 2$, and $y = x^2 - 6x + 10$; about the y-axis



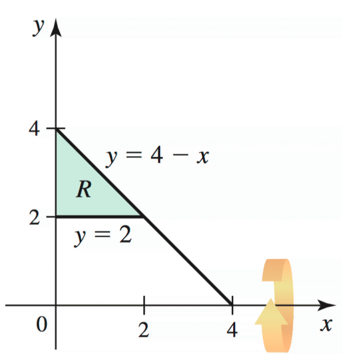
13. $y = \sqrt{x}$, $y = 0$, and $x = 4$; about the x-axis



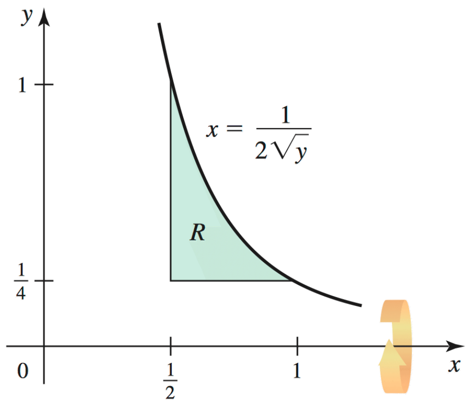
14. $y = \sqrt{x}$, $y = 2 - x$, and $y = 0$; about the x-axis



15. $y = 4 - x$, $y = 2$, and $x = 0$; about the x-axis



16. $x = \dfrac{1}{2\sqrt{y}}$, $x = \dfrac{1}{2}$, and $y = \dfrac{1}{4}$; about the x-axis



17. $y = \cos x^2$ and $y = 0$, for $0 \le x \le \sqrt{\pi/2}$; about the y-axis

18. $y = 6 - x$, $y = 0$, $x = 2$, and $x = 4$; about the y-axis

19. $y = 1 - x^2$, $x = 0$, and $y = 0$, in the first quadrant; about the y-axis

20. $y = \sqrt{x}$, $y = 0$, and $x = 1$; about the y-axis

21. $x = y^2$, $x = 0$, and $y = 3$; about the x-axis

22. $y = x^3$, $y = 1$, and $x = 0$; about the x-axis

23. $y = x$, $y = 2 - x$, and $y = 0$; about the x-axis

24. $y = \sqrt{4 - 2x^2}$, $y = 0$, and $x = 0$, in the first quadrant; about the y-axis

T **25.** $y = \dfrac{1}{(x^2 + 1)^2}$, $y = 0$, $x = 1$, and $x = 2$; about the y-axis

26. $x = y^2$, $x = 4$, and $y = 0$; about the x-axis

T **27.** $y = 2x^{-3/2}$, $y = 2$, $y = 16$, and $x = 0$; about the x-axis

T **28.** $y = (1 - x)^{-1/2}$, $y = 1$, $y = 2$, and $x = 0$; about the y-axis

29. $y = x^3 - x^8 + 1$ and $y = 1$; about the y-axis

T **30.** $y = \sqrt{50 - 2x^2}$, in the first quadrant; about the x-axis

31–34. Shell and washer methods *Let R be the region bounded by the following curves. Use both the shell and washer methods to find the volume of the solid generated when R is revolved about indicated axis.*

31. $y = x$ and $y = x^{1/3}$, in the first quadrant; about the x-axis

32. $y = \dfrac{1}{x+1}$ and $y = 1 - \dfrac{x}{3}$; about the x-axis

33. $y = (x-2)^3 - 2$, $x = 0$, and $y = 25$; about the y-axis

34. $y = 8$, $y = 2x + 2$, $x = 0$, and $x = 2$; about the y-axis

35–40. Shell method about other lines *Let R be the region bounded by $y = x^2$, $x = 1$, and $y = 0$. Use the shell method to find the volume of the solid generated when R is revolved about the following lines.*

35. $x = -2$

36. $x = 1$

37. $x = 2$

38. $y = 1$

39. $y = -2$

40. $y = 2$

41–44. Shell and washer methods about other lines *Use both the shell method and the washer method to find the volume of the solid that is generated when the region in the first quadrant bounded by $y = x^2$, $y = 1$, and $x = 0$ is revolved about the following lines.*

41. $y = -2$

42. $x = -1$

43. $y = 6$

44. $x = 2$

45. **Volume of a sphere** Let R be the region bounded by the upper half of the circle $x^2 + y^2 = r^2$ and the x-axis. A sphere of radius r is obtained by revolving R about the x-axis.

a. Use the shell method to verify that the volume of a sphere of radius r is $\dfrac{4}{3}\pi r^3$.

b. Repeat part (a) using the disk method.

46. **Comparing American and rugby union footballs** An ellipse centered at the origin is described by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are positive constants. If the upper half of such an ellipse is revolved about the x-axis, the resulting surface is an *ellipsoid*.

a. Use the washer method to find the volume of an ellipsoid (in terms of a and b). Check your work using the shell method.

b. Both American and rugby union footballs have the shape of ellipsoids. The maximum regulation size of a rugby union football corresponds to parameters of $a = 6$ in and $b = 3.82$ in and the maximum regulation size of an American football corresponds to parameters of $a = 5.62$ in and $b = 3.38$ in. Find the volume of each football.

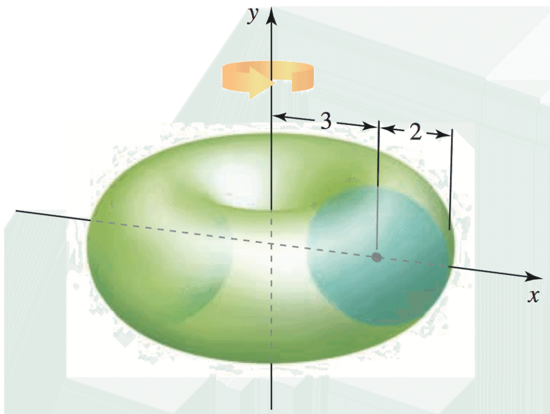
c. Fill in the blank: At their maximum regulation sizes, the volume of a rugby union football has approximately _____ times the volume of an American football.

47. **A torus (doughnut)** A torus is formed when a circle of radius 2 centered at (3, 0) is revolved about the y-axis.

a. Use the shell method to write an integral for volume of the torus.

b. Use the washer method to write an integral for the volume of the torus.

c. Find the volume of the torus by evaluating one of the two integrals obtained in parts (a) and (b). (*Hint:* Both integrals can be evaluated without using the Fundamental Theorem of Calculus.)



48. **A cone by two methods** Verify that the volume of a right circular cone with a base radius of r and a height of h is $\pi r^2 h/3$. Use the region bounded by the line $y = rx/h$, the x-axis, and the line $x = h$, where the region is rotated around the x-axis. Then (a) use the disk method and integrate with respect to x, and (b) use the shell method and integrate with respect to y.

49–58. Choose your method *Let R be the region bounded by the following curves. Use the method of your choice to find the volume of the solid generated when R is revolved about the given axis.*

49. $y = x - x^4$ and $y = 0$; about the x-axis

50. $y = x - x^4$ and $y = 0$; about the y-axis

51. $y = x^2$ and $y = 2 - x^2$; about the x-axis

52. $y = \sin x$ and $y = 1 - \sin x$, for $\pi/6 \le x \le 5\pi/6$; about the x-axis

53. $y = x$, $y = 2x + 2$, $x = 2$, and $x = 6$; about the y-axis

54. $y = x^3$, $y = 0$, and $x = 2$; about the x-axis

55. $y = 4 - x^2$ and $y = 2 - x$, in the first quadrant; about the y-axis

56. $y = 2$, $y = 2x + 2$, and $x = 6$; about the y-axis

57. $y = x^2$, $y = 2 - x$, and $x = 0$, in the first quadrant; about the y-axis

58. $y = \sqrt{x}$, the x-axis, and $x = 4$; about the x-axis

59. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. When using the shell method, the axis of the cylindrical shells is parallel to the axis of revolution.

b. If a region is revolved about the y-axis, then the shell method must be used.

c. If a region is revolved about the x-axis, then in principle, it is possible to use the disk/washer method and integrate with respect to x or the shell method and integrate with respect to y.

Explorations and Challenges »

60–64. Shell method *Use the shell method to find the volume of the following solids.*

60. The solid formed when a hole of radius 2 is drilled symmetrically along the axis of a right circular cylinder of height 6 and radius 4

61. A right circular cone of radius 3 and height 8

62. The solid formed when a hole of radius 3 is drilled symmetrically through the center of a sphere of radius 6

63. The solid formed when a hole of radius 3 is drilled symmetrically along the axis of a right circular cone of radius 6 and height 9

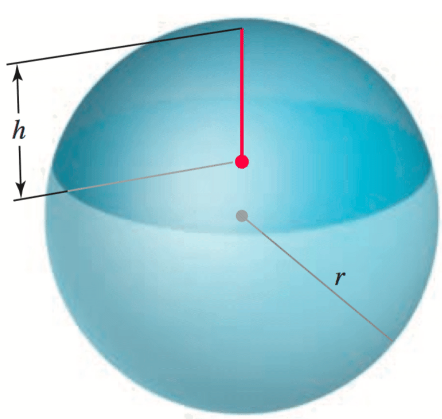
64. A hole of radius $r \le R$ is drilled symmetrically along the axis of a bullet. The bullet is formed by revolving the parabola $y = 6\left(1 - \dfrac{x^2}{R^2}\right)$ about the y-axis, where $0 \le x \le R$.

T **65. Equal volumes** Consider the region R bounded by the curves $y = ax^2 + 1$, $y = 0$, $x = 0$, and $x = 1$, for $a \ge -1$. Let S_1 and S_2 be solids generated when R is revolved about the x- and y-axes, respectively.

a. Find V_1 and V_2, the volumes of S_1 and S_2, as functions of a.

b. What are the values of $a \ge -1$ for which $V_1(a) = V_2(a)$?

66. A spherical cap Consider the cap of thickness h that has been sliced from a sphere of radius r (see figure). Verify that the volume of the cap is $\dfrac{\pi}{3}h^2(3r - h)$ using (a) the washer method, (b) the shell method, and (c) the general slicing method. Check for consistency among the three methods and check the special cases $h = r$ and $h = 0$.



67. Change of variables Suppose $f(x) > 0$, for all x, and $\int_0^4 f(x)\,dx = 10$. Let R be the region in the first quadrant bounded by the coordinate axes, $y = f(x^2)$, and $x = 2$. Find the volume of the solid generated by revolving R around the y-axis.

68. Equal integrals Without evaluating integrals, explain the following equalities. (*Hint:* Draw pictures.)

a. $\pi \int_0^4 (8 - 2x)^2\,dx = 2\pi \int_0^8 y\left(4 - \frac{y}{2}\right) dy$

b. $\int_0^2 \left(25 - \left(x^2 + 1\right)^2\right) dx = 2 \int_1^5 y\sqrt{y - 1}\,dy$

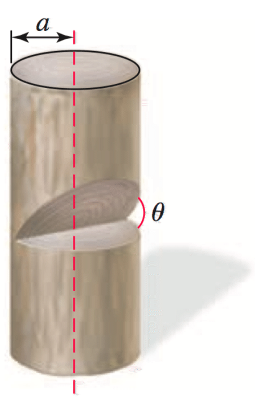
69. Volumes without calculus Solve the following problems with *and* without calculus. A good picture helps.

a. A cube with side length r is inscribed in a sphere, which is inscribed in a right circular cone, which is inscribed in a right circular cylinder. The side length (slant height) of the cone is equal to its diameter. What is the volume of the cylinder?

b. A cube is inscribed in a right circular cone with a radius of 1 and a height of 3. What is the volume of the cube?

c. A cylindrical hole 10 in long is drilled symmetrically through the center of a sphere. How much material is left in the sphere? (Enough information *is* given).

70. Wedge from a tree Imagine a cylindrical tree of radius a. A wedge is cut from the tree by making two cuts: one in a horizontal plane P perpendicular to the axis of the cylinder, and one that makes an angle θ with P, intersecting P along a diameter of the tree (see figure). What is the volume of the wedge?



71. Different axes of revolution Suppose R is the region bounded by $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$, where $f(x) \ge g(x) \ge 0$.

a. Show that if R is revolved about the horizontal line $y = y_0$ that lies below R, then by the washer method, the volume of the resulting solid is

$$V = \int_a^b \pi\left((f(x) - y_0)^2 - (g(x) - y_0)^2\right) dx.$$

b. How is this formula changed if the line $y = y_0$ lies above R?

72. Different axes of revolution Suppose R is the region bounded by $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$, where $f(x) \ge g(x)$.

a. Show that if R is revolved about the vertical line $x = x_0$, where $x_0 < a$, then by the shell method, the volume of the resulting solid is $V = \int_a^b 2\pi (x - x_0)(f(x) - g(x))\,dx$.

b. How is this formula changed if $x_0 > b$?