6.3 Volume by Slicing

We have seen that integration is used to compute the area of two-dimensional regions bounded by curves. Integrals are also used to find the volume of three-dimensional regions (or solids). Once again, the slice-and-sum method is the key to solving these problems.

General Slicing Method »

Consider a solid object that extends in the x-direction from $x = a$ to $x = b$. Imagine making a vertical cut through the solid, perpendicular to the x-axis at a particular point x, and suppose the area of the cross section created by the cut is given by a known integrable function A (**Figure 6.22**).

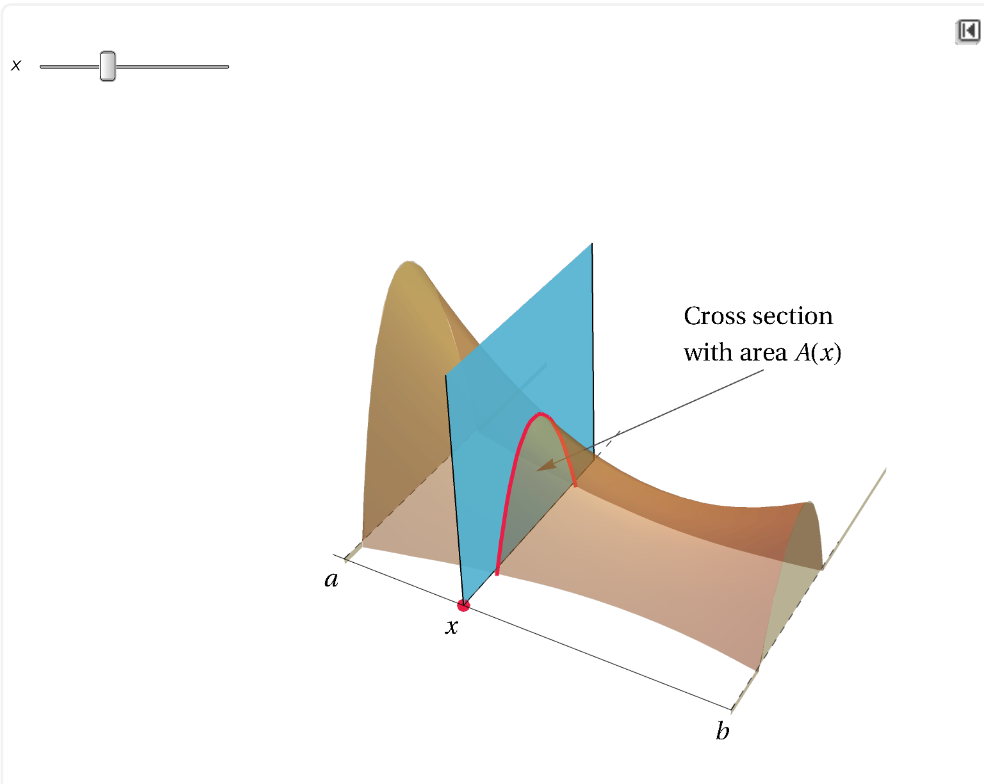


Figure 6.22

To find the volume of this solid, we first divide $[a, b]$ into n subintervals of length $\Delta x = \dfrac{b-a}{n}$. The end-points of the subintervals are the grid points $x_0 = a, x_1, x_2, \ldots, x_n = b$. We now make cuts through the solid perpendicular to the x-axis at each grid point, which produces n slices of thickness Δx. (Imagine cutting a loaf of bread to create n slices of equal width.) On each subinterval, an arbitrary point x_k^* is identified. The kth slice through the solid has a thickness Δx, and we take $A(x_k^*)$ as a representative cross-sectional area of the slice. Therefore, the volume of the kth slice is approximately $A(x_k^*)\,\Delta x$ (**Figure 6.23**). Summing the volumes of the slices, the approximate volume of the solid is

$$V \approx \sum_{k=1}^{n} A(x_k^*)\,\Delta x.$$

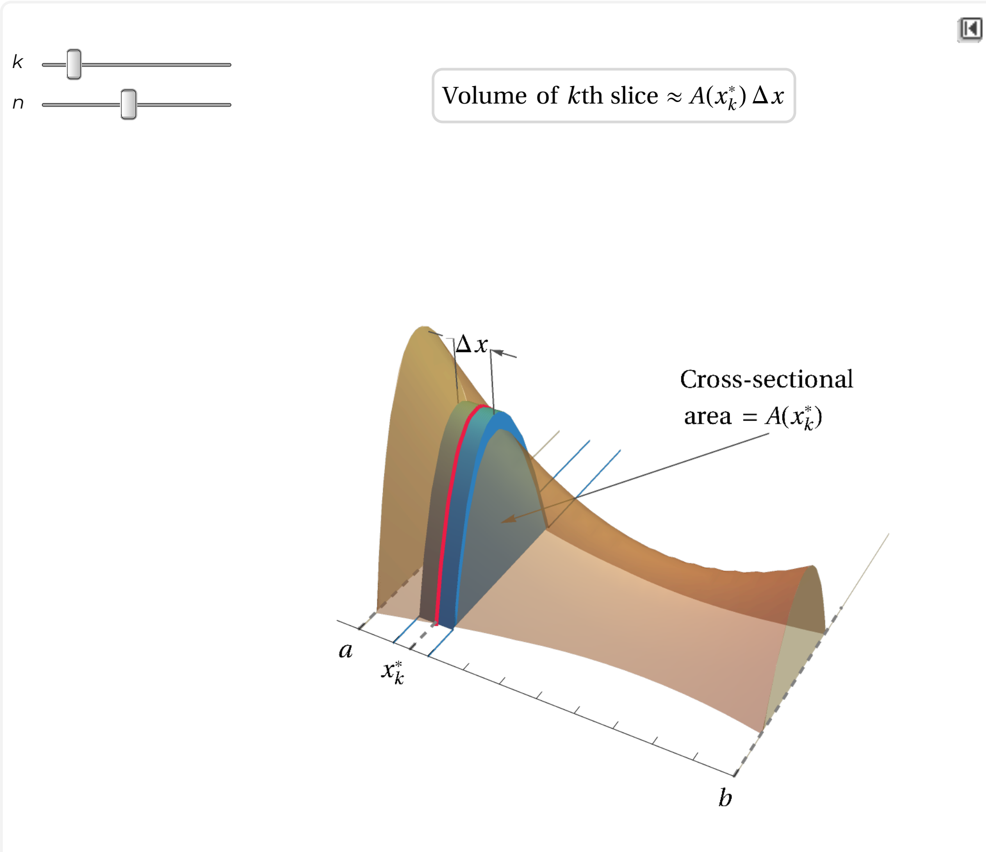


Figure 6.23

As the number of slices increases ($n \to \infty$) and the thickness of each slice approaches zero ($\Delta x \to 0$), the exact volume V is obtained in terms of a definite integral (**Figure 6.24**):

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\,\Delta x = \int_a^b A(x)\,dx.$$

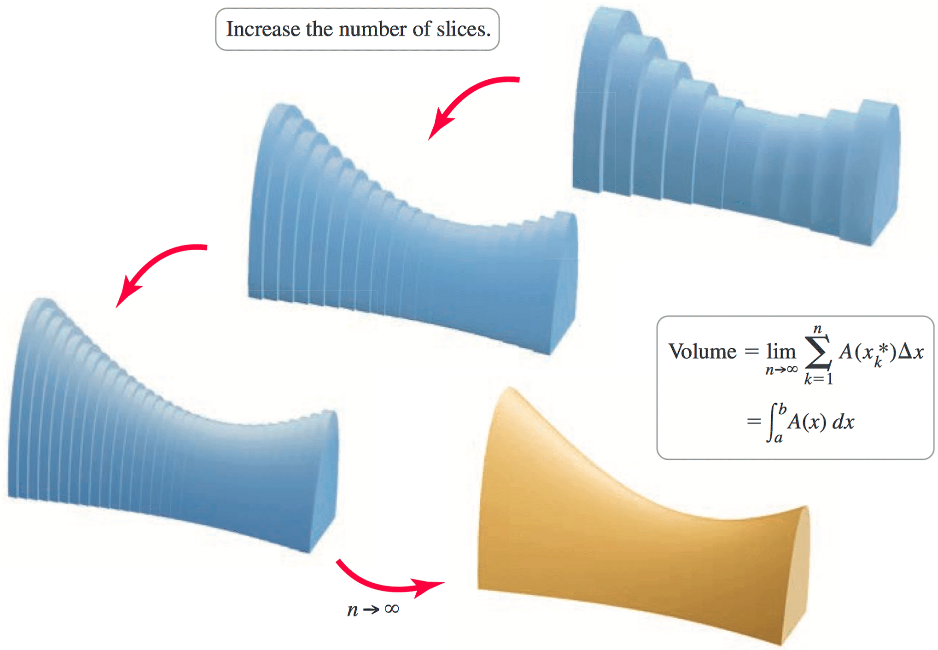


Figure 6.24

We summarize the important general slicing method, which is also the basis of other volume formulas to follow.

General Slicing Method

Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx.$$

Note »

The factors in this volume integral have meaning: $A(x)$ is the cross-sectional area of a slice and dx represents its thickness. Summing (integrating) the volumes of the slices $A(x)\,dx$ gives the volume of the solid.

Quick Check 1 Explain why the volume, as given by the general slicing method, is equal to the average value of $A(x)$ on $[a, b]$ multiplied by $b - a$. ◆

Answer »

The average value of A on $[a, b]$ is $\overline{A} = \frac{1}{b-a}\int_a^b A(x)\,dx$. Therefore, $V = (b-a)\,\overline{A}$.

EXAMPLE 1 Volume of a “parabolic cube”

Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base formed by R, and cross sections through the solid perpendicular to the base and parallel to the y-axis are squares (**Figure 6.25**). Find the volume of the solid.

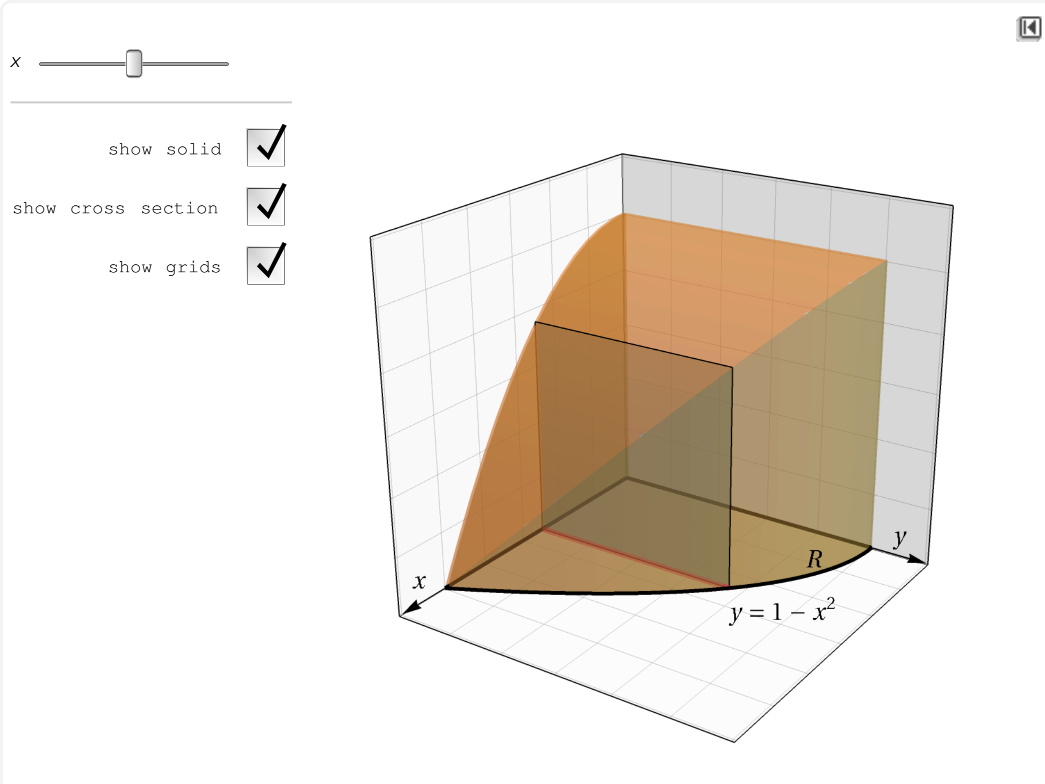


Figure 6.25

SOLUTION »

Focus on a cross section through the solid at a point x, where $0 \le x \le 1$. That cross section is a square with sides of length $1 - x^2$. Therefore, the area of a typical cross section is $A(x) = \left(1 - x^2\right)^2$. Using the general slicing method, the volume of the solid is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx && \text{General slicing method} \\ &= \int_0^1 \left(1 - x^2\right)^2 dx && \text{Substitute for } A(x). \\ &= \int_0^1 \left(1 - 2x^2 - x^4\right) dx && \text{Expand integrand.} \\ &= \frac{8}{15}. && \text{Evaluate.} \end{aligned}$$

The actual solid with its square cross sections is shown in Figure 6.25.

Related Exercises 11–12 ◆

EXAMPLE 2 Volume of a "parabolic hemisphere"

A solid has a base that is bounded by the curves $y = x^2$ and $y = 2 - x^2$ in the xy-plane. Cross sections through the solid perpendicular to the base and parallel to the y-axis are semicircular disks. Find the volume of the solid.

SOLUTION »

Because a typical cross section perpendicular to the x-axis is a semicircular disk (**Figure 6.26**), the area of a cross section is $\frac{1}{2}\pi r^2$, where r is the radius of the cross section. The key observation is that this radius is onehalf of the distance between the upper bounding curve $y = 2 - x^2$ and the lower bounding curve $y = x^2$. So the radius at the point x is

$$r = \frac{1}{2}\left(\left(2 - x^2\right) - x^2\right) = 1 - x^2.$$

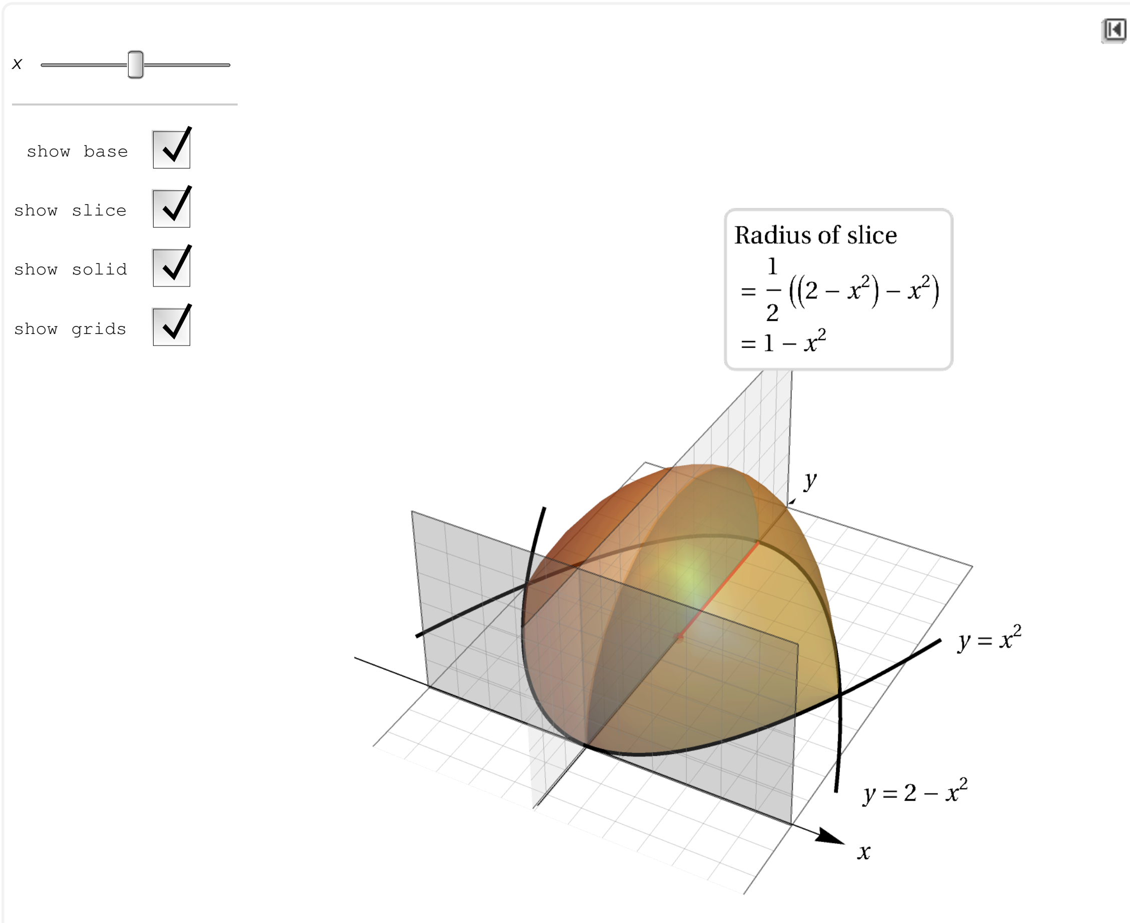


Figure 6.26

This means that the area of the semicircular cross section at the point x is

$$A(x) = \frac{1}{2}\pi\, r^2 = \frac{\pi}{2}\left(1 - x^2\right)^2.$$

The intersection points of the two bounding curves satisfy $2 - x^2 = x^2$, which has solutions $x = \pm 1$. Therefore, the cross sections lie between $x = -1$ and $x = 1$. Integrating the cross-sectional areas, the volume of the solid is

$$
\begin{aligned}
V &= \int_{-1}^{1} A(x)\,dx && \text{General slicing method} \\
&= \int_{-1}^{1} \frac{\pi}{2}\left(1 - x^2\right)^2 dx && \text{Substitute for } A(x). \\
&= \frac{\pi}{2}\int_{-1}^{1}\left(1 - 2x^2 + x^4\right) dx && \text{Expand integrand.} \\
&= \frac{8\pi}{15}. && \text{Evaluate.}
\end{aligned}
$$

Related Exercise 15 ◆

Quick Check 2 In Example 2, what is the cross-sectional area function $A(x)$ if cross sections perpendicular to the base are squares rather than semicircles? ◆

Answer »

$A(x) = \left(2 - 2x^2\right)^2$

The Disk Method »

We now consider a specific type of solid known as a **solid of revolution**. Suppose f is a continuous function with $f(x) \ge 0$ on an interval $[a, b]$. Let R be the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ (**Figure 6.27**).

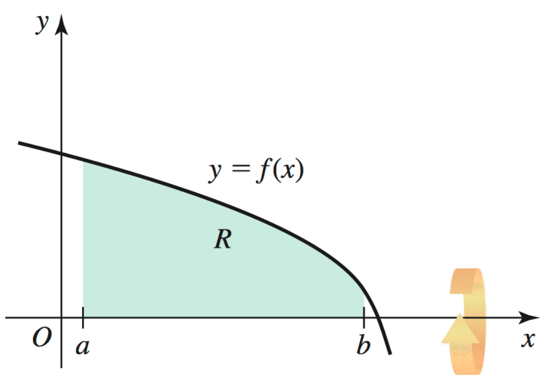


Figure 6.27

Now revolve R around the x-axis. As R revolves once around the x-axis, it sweeps out a three-dimensional solid of revolution (**Figure 6.28**). The goal is to find the volume of this solid, which may be done using the general slicing method.

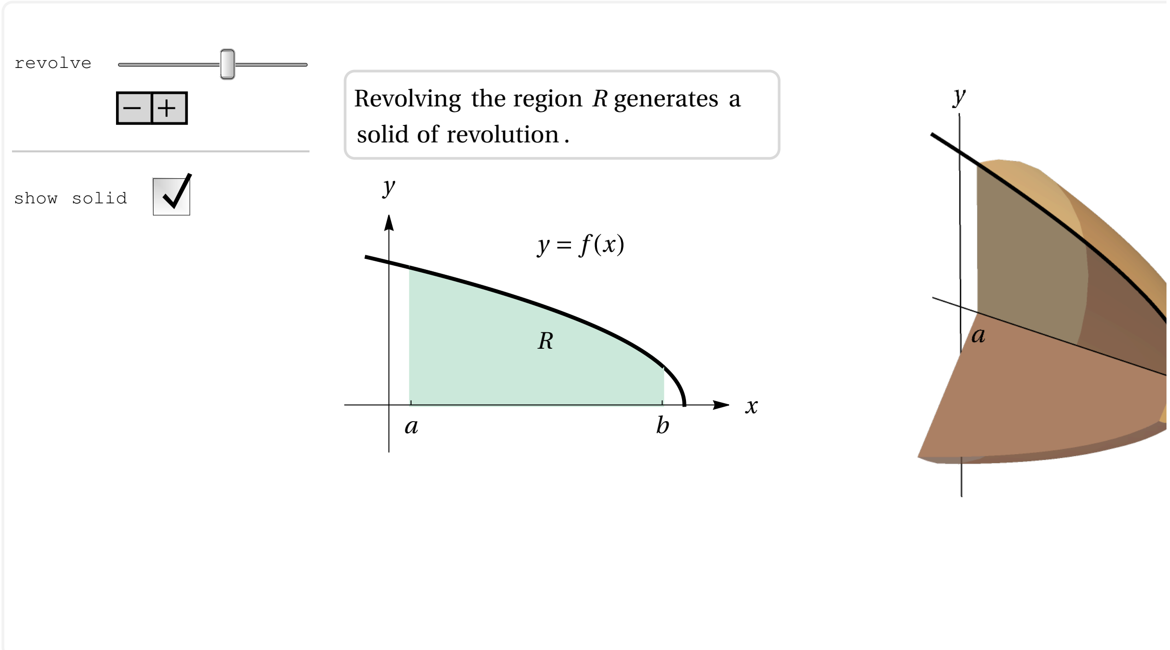


Figure 6.28

Quick Check 3 What solid results when the region R is revolved about the x-axis if (a) R is a square with vertices (0, 0), (0, 2), (2, 0), and (2, 2) and (b) R is a triangle with vertices (0, 0), (0, 2), and (2, 0)? ◆

Answer »

(a) A cylinder with height 2 and radius 2; (b) a cone with height 2 and base radius 2

With a solid of revolution, the cross-sectional area function has a special form because all cross sections perpendicular to the x-axis are *circular disks* with radius $f(x)$ (**Figure 6.29**). Therefore, the cross section at the point x, where $a \le x \le b$, has area

$$A(x) = \pi\,(\text{radius})^2 = \pi\, f(x)^2.$$

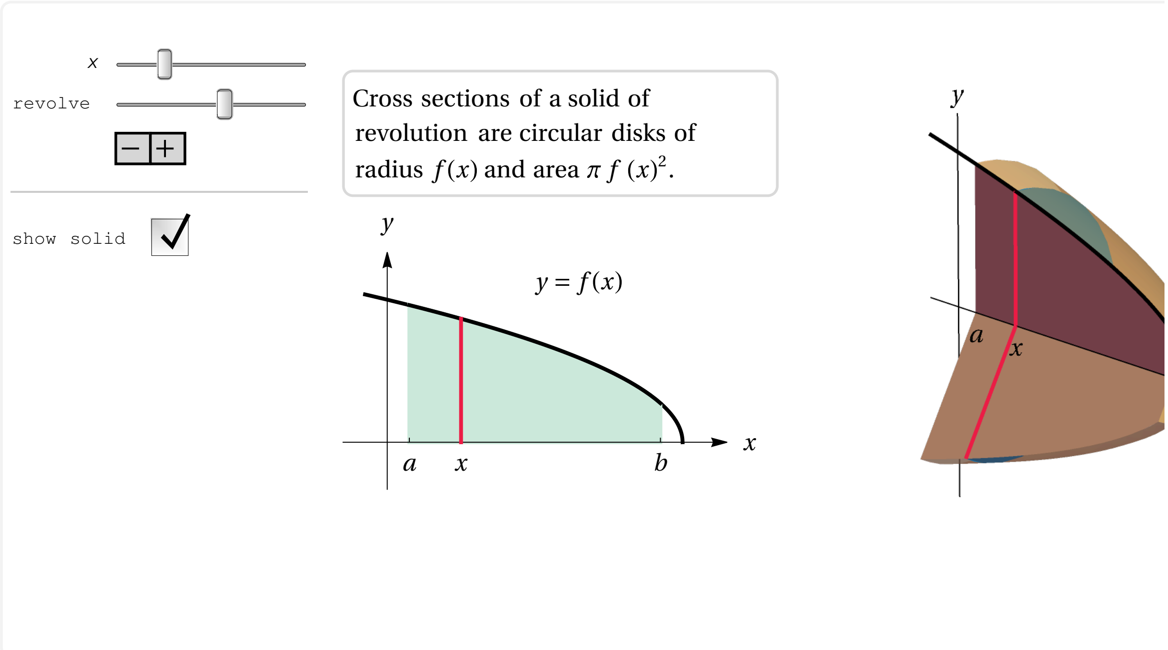


Figure 6.29

By the general slicing method, the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi f(x)^2\,dx.$$

Because each slice through the solid is a circular disk, the resulting method is called the *disk method*.

Disk Method about the x-Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \underbrace{f(x)}_{\text{disk radius}}{}^2\,dx.$$

EXAMPLE 3 Disk method at work

Let R be the region bounded by the curve $f(x) = (x+1)^2$, the x-axis, and the lines $x = 0$ and $x = 2$. Find the volume of the solid of revolution obtained by revolving R about the x-axis.

SOLUTION »

When the region R is revolved about the x-axis, it generates a solid of revolution (**Figure 6.30**). A cross section perpendicular to the x-axis at the point $0 \leq x \leq 2$ is a circular disk of radius $f(x)$. Therefore, a typical cross section has area

$$A(x) = \pi f(x)^2 = \pi\left((x+1)^2\right)^2.$$

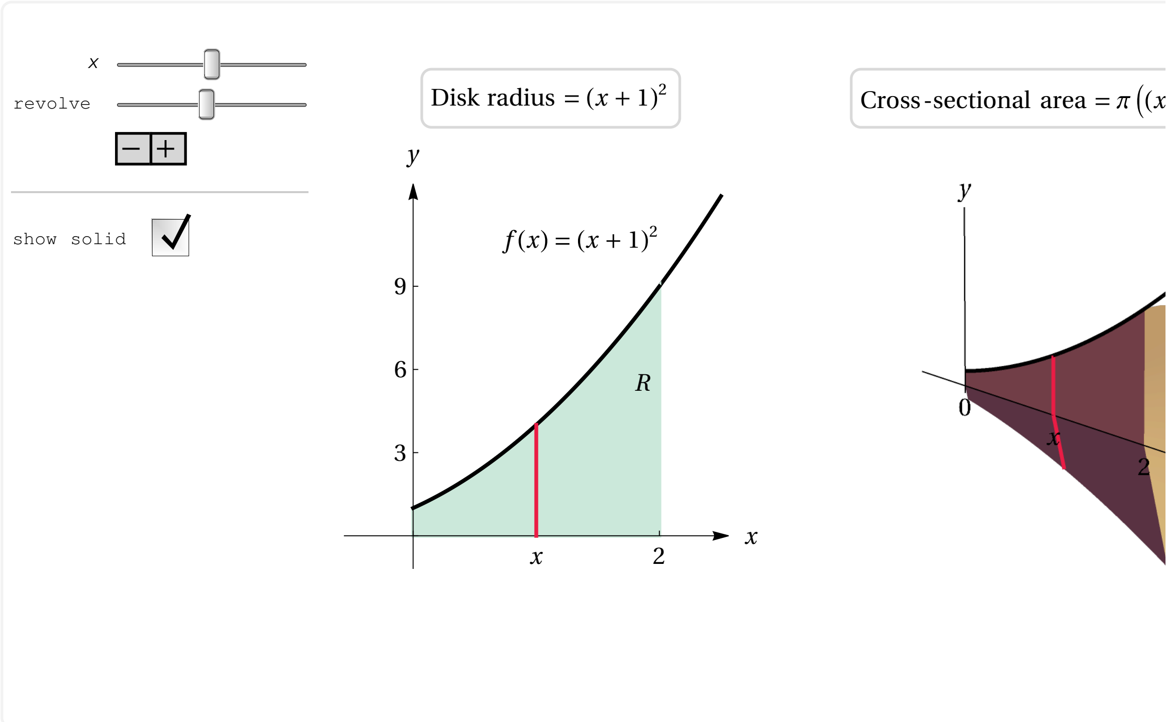


Figure 6.30

Integrating these cross-sectional areas between $x = 0$ and $x = 2$ gives the volume of the solid:

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi\,(\underbrace{(x+1)^2}_{\text{disk radius}})^2\,dx \quad \text{Substitute for } A(x).$$

$$= \int_0^2 \pi\,(x+1)^4\,dx \quad \text{Simplify.}$$

$$= \pi \left.\frac{u^5}{5}\right|_1^3 = \frac{242\,\pi}{5}. \quad \text{Let } u = x+1 \text{ and evaluate.}$$

Related Exercises 17, 19 ◆

Washer Method

A slight variation on the disk method enables us to compute the volume of more exotic solids of revolution. Suppose R is the region bounded by the graphs of f and g between $x = a$ and $x = b$, where $f(x) \geq g(x) \geq 0$ (**Figure 6.31**). If R is revolved about the x-axis to generate a solid of revolution, the resulting solid generally has a hole through it.

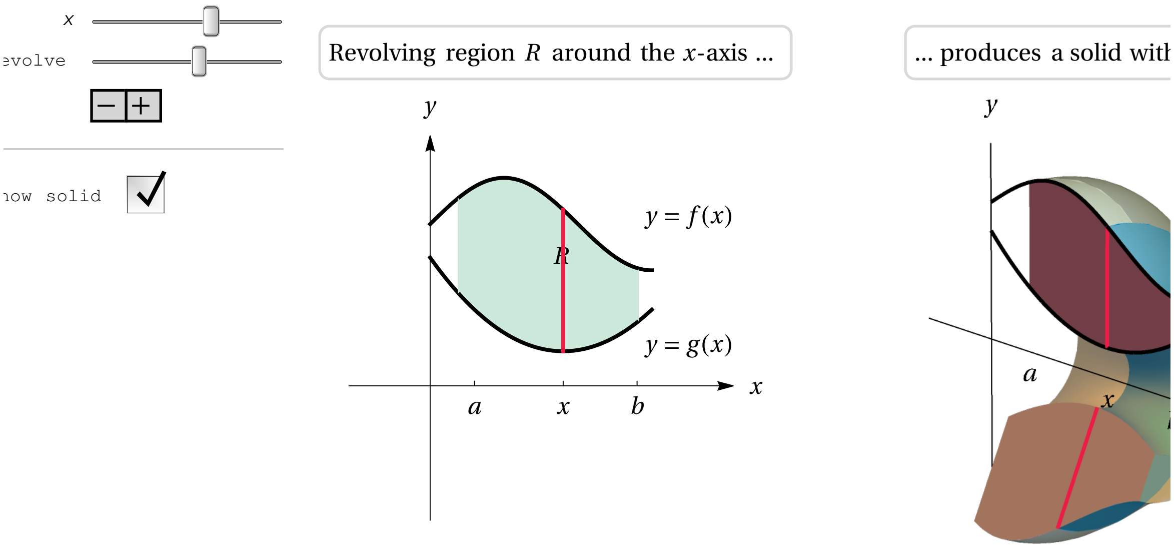


Figure 6.31

Once again we apply the general slicing method. In this case, a cross section through the solid perpendicular to the x-axis is a circular *washer* with an outer radius of $r_o = f(x)$ and an inner radius of $r_i = g(x)$, where $a \le x \le b$. The area of the cross section is the area of the entire disk minus the area of the hole, or

$$A(x) = \pi\left(r_o^{\,2} - r_i^{\,2}\right) = \pi\left(f(x)^2 - g(x)^2\right)$$

(**Figure 6.32**). The general slicing method gives the volume of the solid.

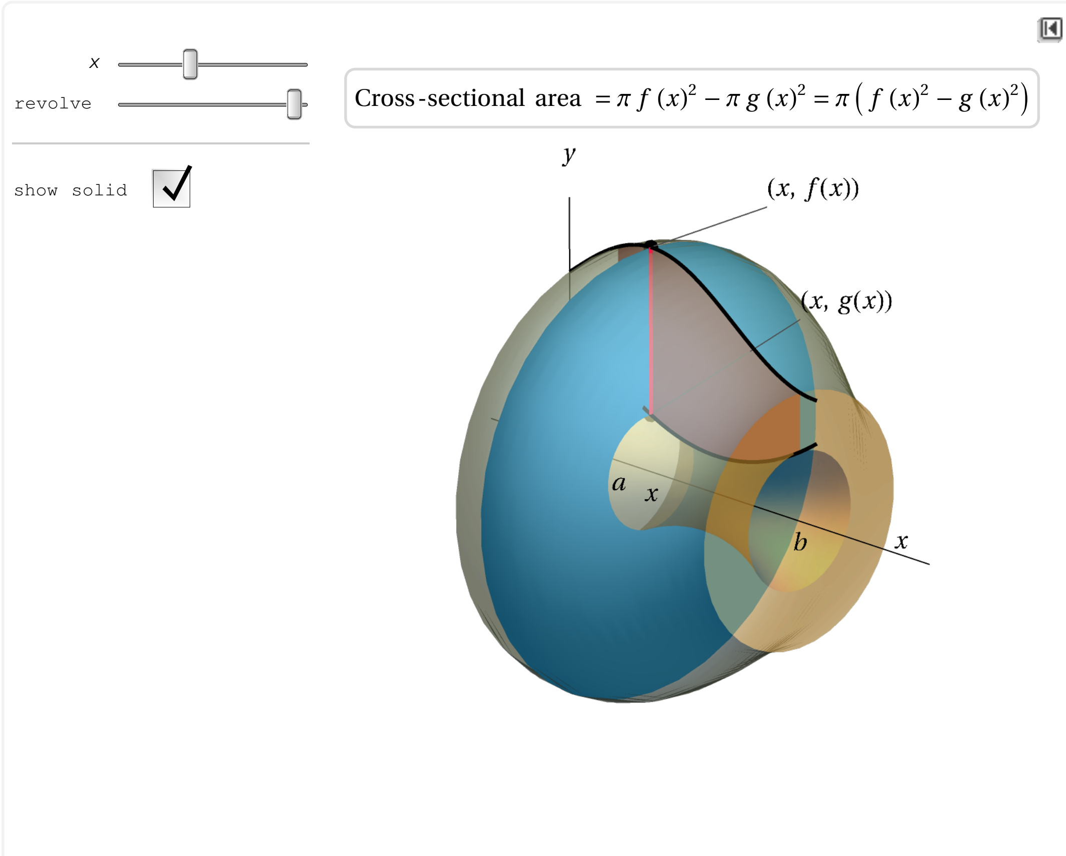


Figure 6.32

Washer Method about the x-Axis

Let f and g be continuous functions with $f(x) \ge g(x) \ge 0$ on $[a, b]$. Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \left(\pi \underbrace{f(x)}_{\substack{\text{outer}\\ \text{radius}}}{}^2 - \pi \underbrace{g(x)}_{\substack{\text{inner}\\ \text{radius}}}{}^2\right) dx.$$

Note »

The washer method is really two applications of the disk method. We compute the volume of the entire solid without the hole (by the disk method) and then subtract the volume of the hole (also computed by the disk method).

Quick Check 4 Show that when $g(x) = 0$ in the washer method, the result is the disk method. ◆

Answer »

EXAMPLE 4 Volume by the washer method

The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results when R is revolved about the x-axis?

SOLUTION »

The region R is bounded by the graphs of f and g with $f(x) \geq g(x)$ on $[0, 1]$, so the washer method is applicable (**Figure 6.33**). The area of a typical cross section at the point x is

$$A(x) = \pi f(x)^2 - \pi g(x)^2 = \pi \underbrace{\left(\sqrt{x}\right)}_{\text{outer radius}}^2 - \pi \underbrace{\left(x^2\right)}_{\text{inner radius}}^2 = \pi x - \pi x^4.$$

Note »

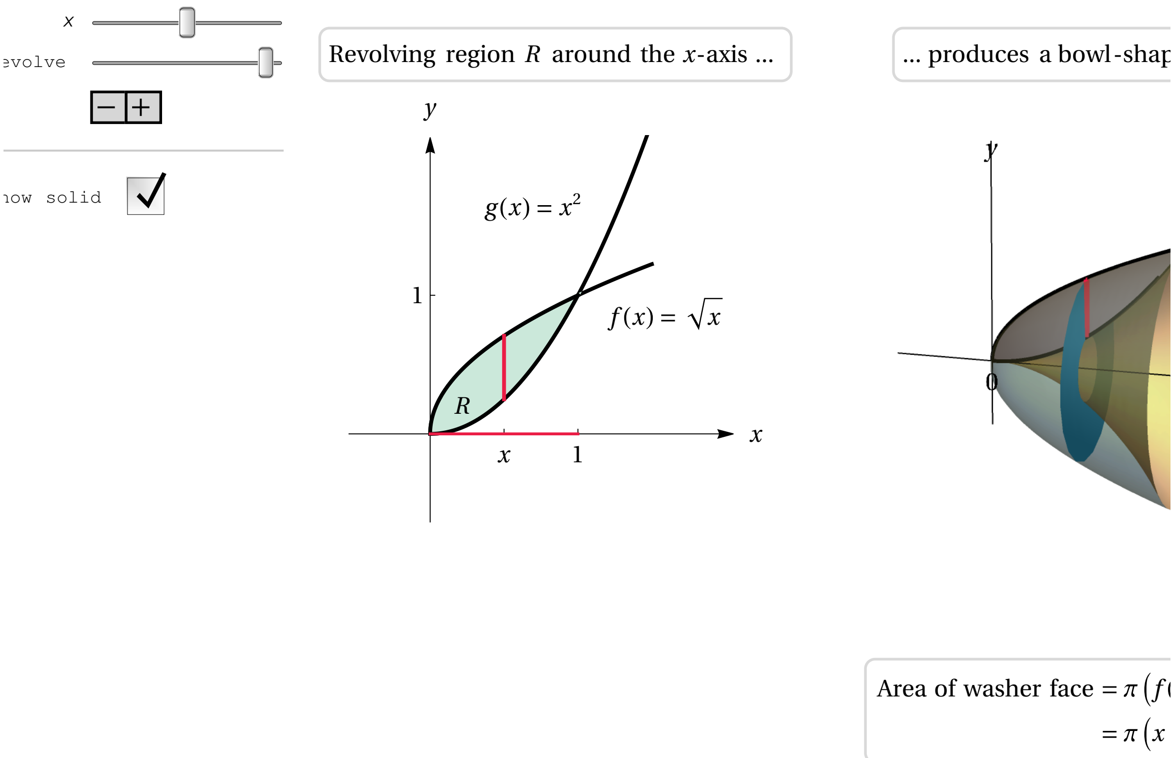


Figure 6.33

Therefore, the volume of the solid is

$$\begin{aligned} V &= \int_0^1 \pi \left(x - x^4\right) dx && \text{Washer method} \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3\pi}{10}. && \text{Fundamental Theorem} \end{aligned}$$

Related Exercises 21–22 ◆

Quick Check 5 Suppose the region in Example 4 is revolved about the line $y = -1$ instead of the x-axis. (a) What is the inner radius of a typical washer? (b) What is the outer radius of a typical washer? ◆

Answer »

(a) Inner radius $= x^2 + 1$; (b) outer radius $= \sqrt{x} + 1$

Revolving About the *y*-Axis »

Everything you learned about revolving regions about the x-axis applies to revolving regions about the y-axis. Consider a region R bounded by the curve $x = p(y)$ on the right, the curve $x = q(y)$ on the left, and the horizontal lines $y = c$ and $y = d$ (**Figure 6.34**).

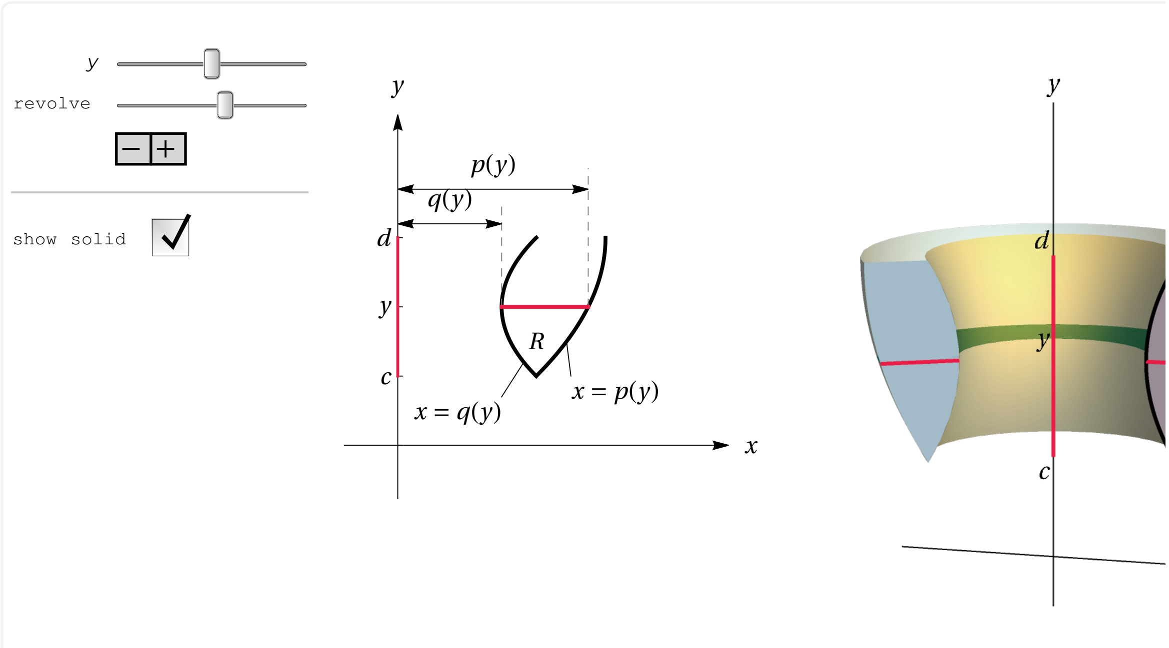


Figure 6.34

To find the volume of the solid generated when R is revolved about the y-axis, we use the general slicing method—now with respect to the y-axis. The area of a typical cross section is $A(y) = \pi\left(p(y)^2 - q(y)^2\right)$, where $c \le y \le d$. As before, integrating these cross-sectional areas of the solid gives the volume.

Disk and Washer Methods About the y-Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by the curves $x = p(y)$ and $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi \underbrace{p(y)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \pi \underbrace{q(y)}_{\substack{\text{inner}\\\text{radius}}}{}^2 \, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi \underbrace{p(y)}_{\substack{\text{disk}\\\text{radius}}}{}^2 \, dy.$$

Note »

The disk/washer method about the y-axis is the disk/washer method about the x-axis with x replaced by y.

EXAMPLE 5 Which solid has greater volume?

Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

SOLUTION »

Solving $y^3 = 4y$ or, equivalently, $y\left(y^2 - 4\right) = 0$, we find that the bounding curves of R intersect at the points $(0, 0)$ and $(8, 2)$. When the region R is revolved about the y-axis, it generates a funnel with a curved inner surface (**Figure 6.35**).

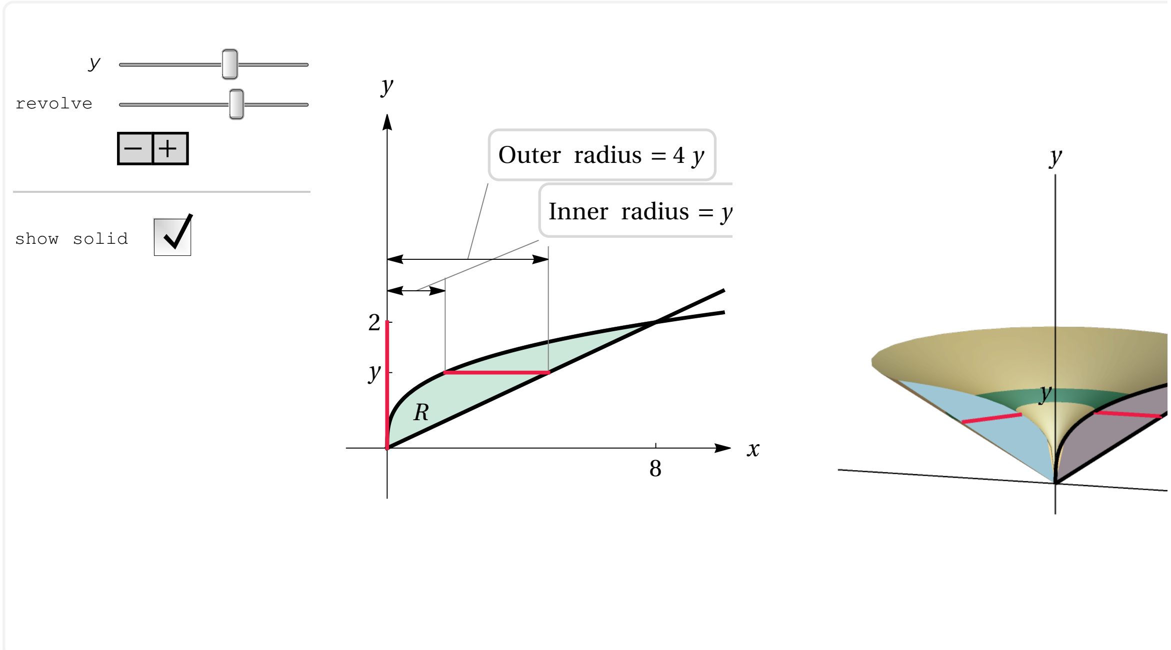


Figure 6.35

Washer-shaped cross sections perpendicular to the y-axis extend from $y = 0$ to $y = 2$. The outer radius of the cross section at the point y is determined by the line $x = p(y) = 4y$. The inner radius of the cross section at the point y is determined by the curve $x = q(y) = y^3$. Applying the washer method, the volume of this solid is

$$\begin{aligned} V &= \int_0^2 \pi\left(p(y)^2 - q(y)^2\right) dy && \text{Washer method} \\ &= \int_0^2 \pi\left(16y^2 - y^6\right) dy && \text{Substitute for } p \text{ and } q. \\ &= \pi\left(\frac{16y^3}{3} - \frac{y^7}{7}\right)\Bigg|_0^2 && \text{Fundamental Theorem} \\ &= \frac{512\pi}{21}. && \text{Evaluate.} \end{aligned}$$

When the region R is revolved about the x-axis, it generates a different funnel with a flat inner surface (**Figure 6.36**).

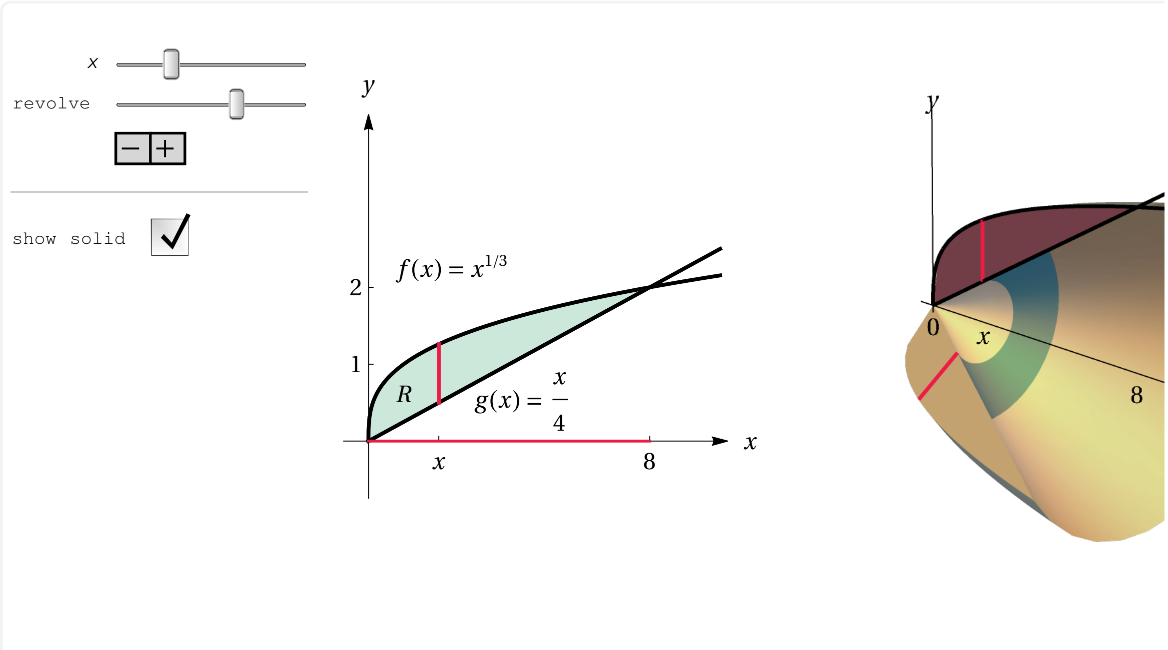


Figure 6.36

Vertical slices through the solid between $x = 0$ and $x = 8$ produce washers. The outer radius of the washer at the point x is determined by the curve $x = y^3$, or $y = f(x) = x^{1/3}$. The inner radius is determined by $x = 4y$, or $y = g(x) = \frac{x}{4}$. The volume of the resulting solid is

$$\begin{aligned} V &= \int_0^8 \pi\left(f(x)^2 - g(x)^2\right) dx && \text{Washer method} \\ &= \int_0^8 \pi\left(x^{2/3} - \frac{x^2}{16}\right) dx && \text{Substitute for } f \text{ and } g. \\ &= \pi\left(\frac{3}{5}x^{5/3} - \frac{x^3}{48}\right)\Bigg|_0^8 && \text{Fundamental Theorem} \\ &= \frac{128\pi}{15}. && \text{Evaluate.} \end{aligned}$$

We see that revolving the region about the y-axis produces a solid of greater volume.

Related Exercises 43–44 ◆

Quick Check 6 The region in the first quadrant bounded by $y = x$ and $y = x^3$ is revolved about the y-axis. Give the integral for the volume of the solid that is generated. ◆

Answer »

$$\int_0^1 \pi\left(y^{2/3} - y^2\right) dy$$

The disk and washer methods may be generalized to handle situations where a region R is revolved about a line parallel to one of the coordinate axes. The next example discusses three such cases.

EXAMPLE 6 **Revolving about other lines**

Let $f(x) = \sqrt{x} + 1$ and $g(x) = x^2 + 1$.

a. Find the volume of the solid generated when the region R_1 bounded by the graph of f and the line $y = 2$ on the interval $[0, 1]$ is revolved about the line $y = 2$.

b. Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0, 1]$ is revolved about the line $y = -1$.

c. Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0, 1]$ is revolved about the line $x = 2$.

SOLUTION »

a. **Figure 6.37a** shows the region R_1 and the axis of revolution. Applying the disk method, we see that a disk located at a point x has a radius of $2 - f(x) = 2 - \left(\sqrt{x} + 1\right) = 1 - \sqrt{x}$. Therefore, the volume of the solid generated when R_1 is revolved about $y = 2$ is

$$\int_0^1 \pi \underbrace{\left(1 - \sqrt{x}\right)^2}_{\text{disk radius}} dx = \pi \int_0^1 \left(1 - 2\sqrt{x} + x\right) dx = \frac{\pi}{6}.$$

b. When the graph of f is revolved about $y = -1$, it sweeps out a solid of revolution whose radius at a point x is $f(x) + 1 = \sqrt{x} + 2$. Similarly, when the graph of g is revolved about $y = -1$, it sweeps out a solid of revolution whose radius at a point x is $g(x) + 1 = x^2 + 2$ (**Figure 6.37b**). Using the washer method, the volume of the solid generated when R_2 is revolved about $y = -1$ is

$$\int_0^1 \pi \underbrace{\left(\sqrt{x} + 2\right)^2}_{\text{outer radius}} - \pi \underbrace{\left(x^2 + 2\right)^2}_{\text{inner radius}} dx = \pi \int_0^1 \left(-x^4 - 4x^2 + x + 4\sqrt{x}\right) dx \quad \text{Washer method; Simplify integrand.}$$

$$= \frac{49\pi}{30}. \quad \text{Evaluate integral.}$$

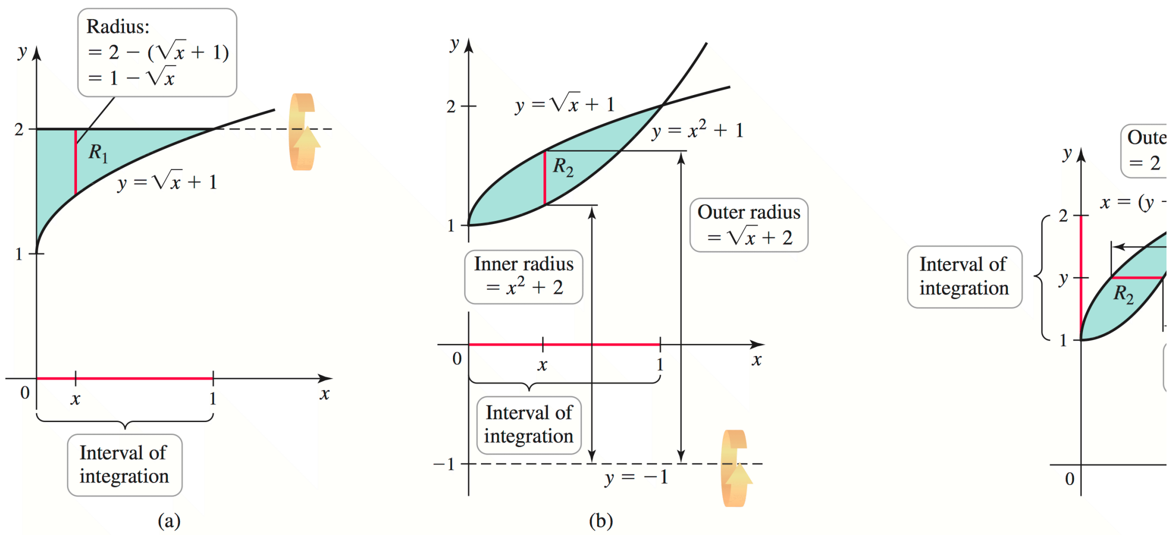


Figure 6.37

c. When the region R_2 is revolved about the line $x = 2$, we use the washer method and integrate in the y-direction. First note that the graph of f is described by $y = \sqrt{x} + 1$, or equivalently, $x = (y-1)^2$, for $y \geq 1$. Also, the graph of g is described by $y = x^2 + 1$, or equivalently, $x = \sqrt{y-1}$ for $y \geq 1$ (**Figure 6.37c**). When the graph of f is revolved about the line $x = 2$, the radius of a typical disk at a point y is $2 - (y-1)^2$. Similarly, when the graph of g is revolved about $x = 2$, the radius of a typical disk at a point y is $2 - \sqrt{y-1}$. Finally, observe that the extent of the region R_2 in the y-direction is the interval $1 \leq y \leq 2$.

Applying the washer method, simplifying the integrand, and integrating powers of y, the volume of the solid of revolution is

$$\int_1^2 \pi \underbrace{\left(2-(y-1)^2\right)}_{\text{outer radius}}{}^2 - \pi \underbrace{\left(2-\sqrt{y-1}\right)}_{\text{inner radius}}{}^2 \, dy = \frac{31\pi}{30}.$$

Related Exercises 45, 48 ◆

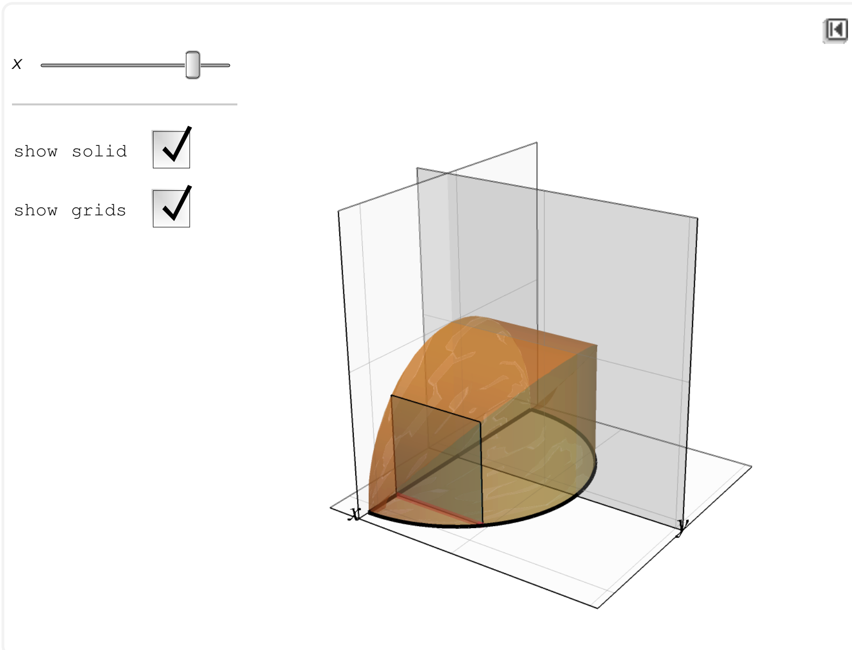
Exercises »

Getting Started »

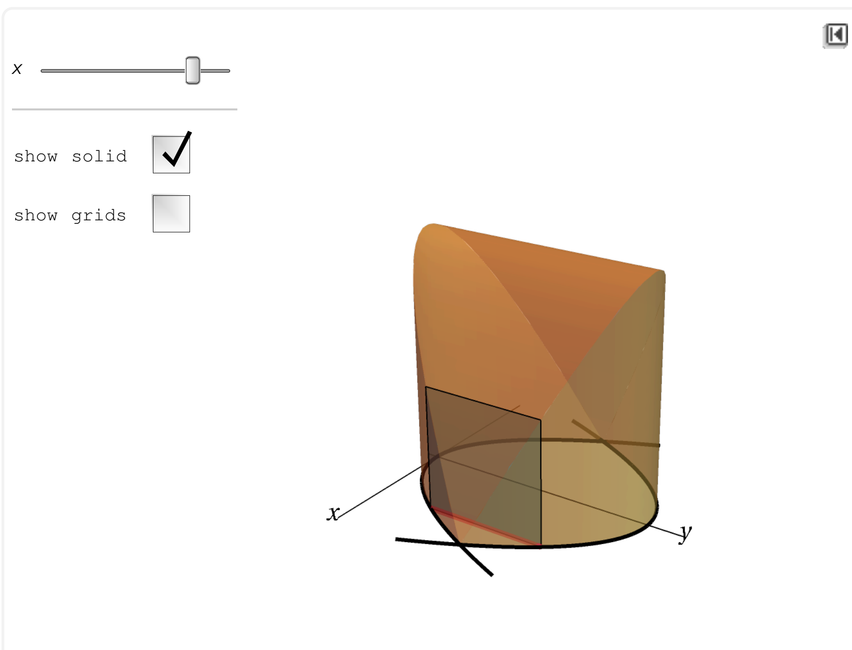
Practice Exercises »

11–16. General slicing method *Use the general slicing method to find the volume of the following solids.*

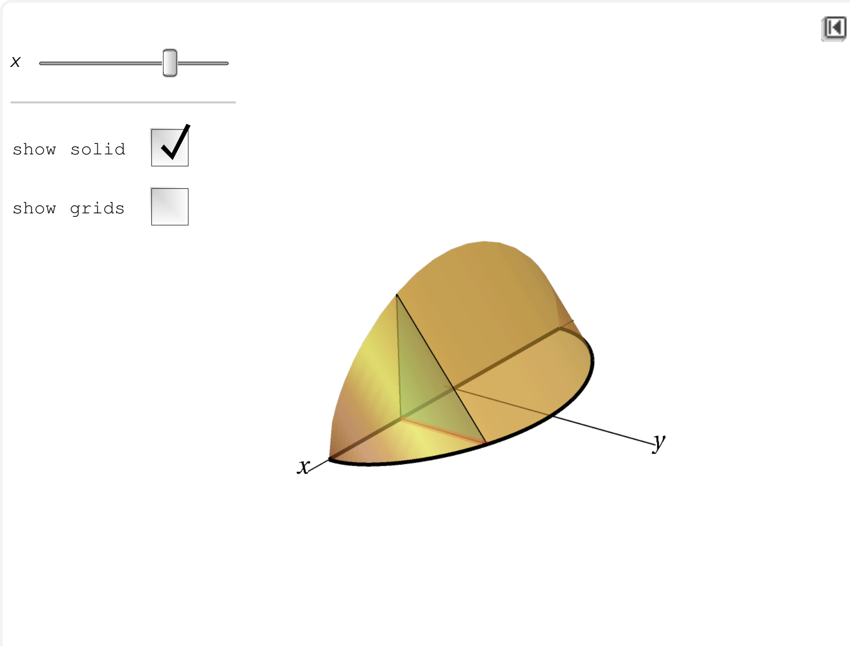
11. The solid whose base is the region bounded by the semicircle $y = \sqrt{1-x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares



12. The solid whose base is the region bounded by the curves $y = x^2$ and $y = 2 - x^2$, and whose cross sections through the solid perpendicular to the x-axis are squares



13. The solid whose base is the region bounded by the curve $y = \sqrt{\cos x}$ and the x-axis on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and whose cross sections through the solid perpendicular to the x-axis are isosceles right triangles with a horizontal leg in the xy-plane and a vertical leg above the x-axis



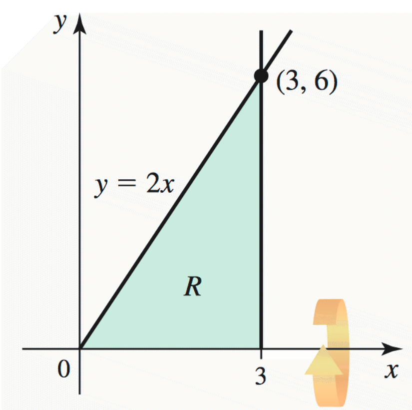
14. The solid with a semicircular base of radius 5 whose cross sections perpendicular to the base and parallel to the diameter are squares

15. The solid whose base is the triangle with vertices (0, 0), (2, 0), and (0, 2), and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles

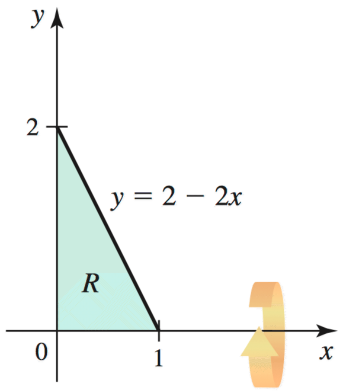
16. The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$, and whose cross sections perpendicular to the base and parallel to the y-axis are squares

17–40. Solids of revolution *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given axis.*

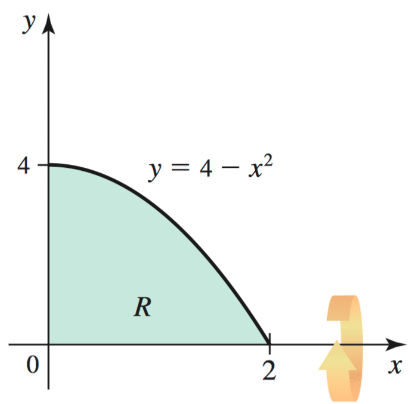
17. $y = 2x$, $y = 0$, and $x = 3$; about the x-axis (Verify that your answer agrees with the volume formula for a cone.)



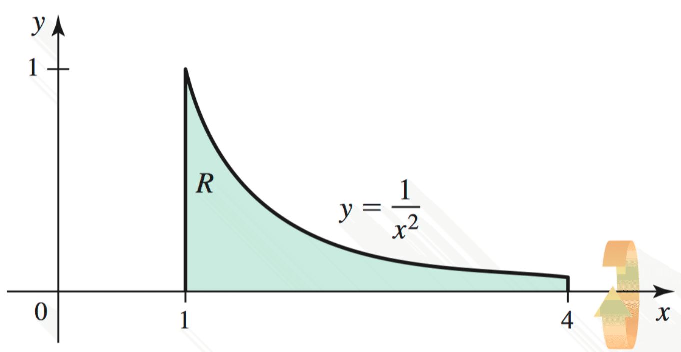
18. $y = 2 - 2x$, $y = 0$, and $x = 0$; about the x-axis (Verify that your answer agrees with the volume formula for a cone.)



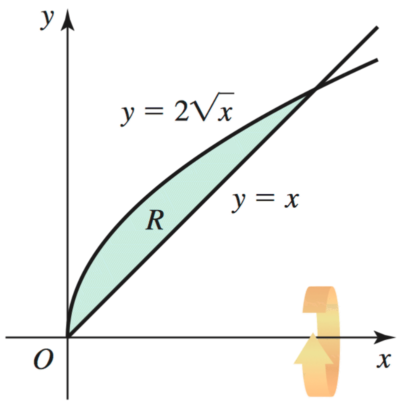
19. $y = 4 - x^2$, $y = 0$, and $x = 0$; about the x-axis



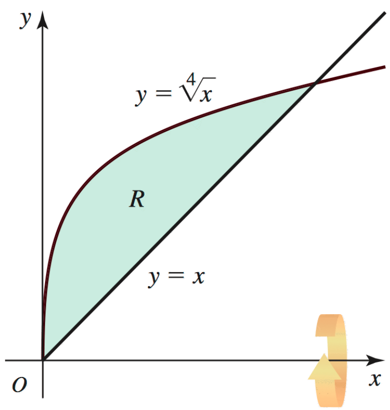
20. $y = \frac{1}{x^2}$, $y = 0$, $x = 1$, and $x = 4$; about the x-axis



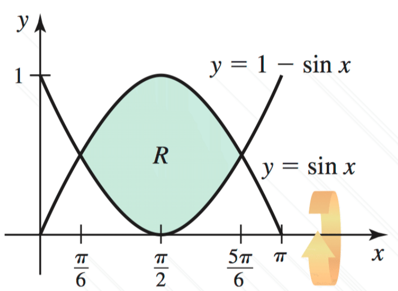
21. $y = x$ and $y = 2\sqrt{x}$; about the x-axis



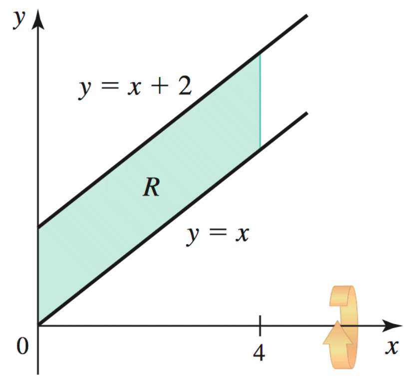
22. $y = x$ and $y = \sqrt[4]{x}$; about the x-axis



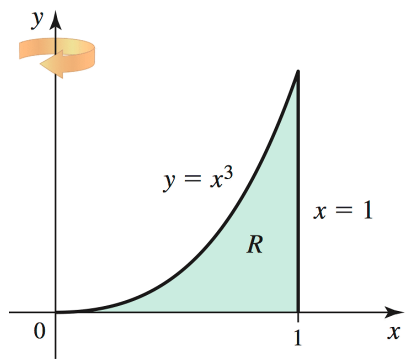
23. $y = \sin x$, $y = 1 - \sin x$, $x = \pi/6$, and $x = 5\pi/6$; about the x-axis



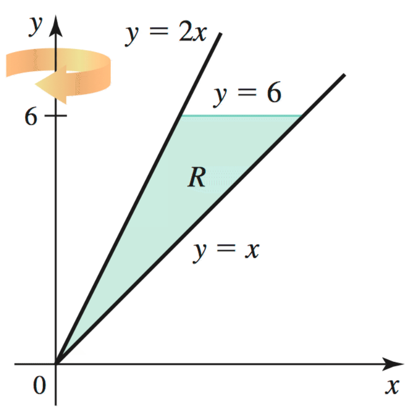
24. $y = x$, $y = x + 2$, $x = 0$, and $x = 4$; about the x-axis



25. $y = x^3$, $y = 0$, and $x = 1$; about the y-axis



26. $y = x$, $y = 2x$, and $y = 6$; about the y-axis



27. $y = \cos x$ on $\left[0, \dfrac{\pi}{2}\right]$, $y = 0$, and $x = 0$; about the x-axis (*Hint*: Recall that $\cos^2 x = \dfrac{1 + \cos 2x}{2}$.)

28. $y = \sqrt{25 - x^2}$ and $y = 0$; about the x-axis (Verify that your answer agrees with the volume formula for a sphere.)

29. $y = \sin x$ on $[0, \pi]$ and $y = 0$; about the x-axis (*Hint:* Recall that $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.)

30. $y = \sec x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{4}$; about the x-axis

31. $y = 2 + \sqrt{x}$, $y = 2 - \sqrt{x}$, and $x = 4$; about the x-axis

32. $y = \sqrt{\sin x}$, $y = 1$, and $x = 0$; about the x-axis

33. $y = \sin x$ and $y = \sqrt{\sin x}$, for $0 \le x \le \frac{\pi}{2}$; about the x-axis

34. $y = |x|$ and $y = 2 - x^2$; about the x-axis

35. $y = \sqrt{x}$, $y = 0$, and $x = 4$; about the y-axis

36. $y = 4 - x^2$, $x = 2$, and $y = 4$; about the y-axis

37. $y = x^{-3/2}$, $y = 1$, $x = 1$, and $x = 6$; about the x-axis

38. $y = x^2$, $y = 2 - x$, and $y = 0$; about the y-axis

39. $y = x^{1/3}$, $y = 4 - x^{1/3}$, and $x = 0$; about the y-axis.

40. $y = x^{-1}$, $y = 0$, $x = 1$, and $x = p > 0$; about the x-axis (Is the volume bounded as $p \to \infty$?)

41–44. Which is greater? *For the following regions R, determine which is greater: the volume of the solid generated when R is revolved about the x-axis or about the y-axis.*

41. R is bounded by $y = 2x$, the x-axis, and $x = 5$.

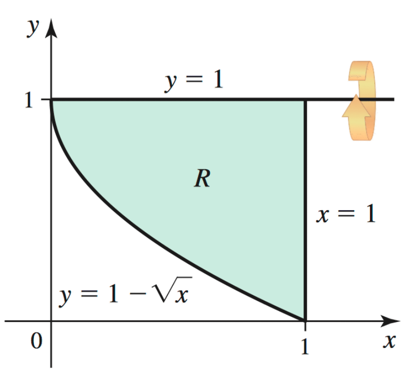
42. R is bounded by $y = 4 - 2x$, the x-axis, and the y-axis.

43. R is bounded by $y = 1 - x^3$, the x-axis, and the y-axis.

44. R is bounded by $y = x^2$ and $y = \sqrt{8x}$.

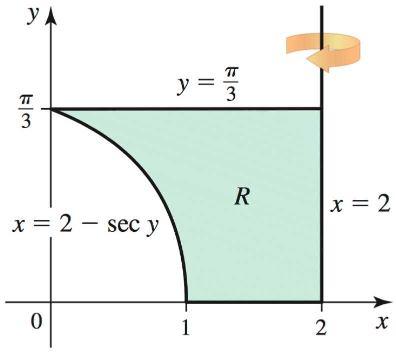
45–55. Revolution about other axes *Let R be the region bounded by the following curves. Find the volume of the solid generated when R is revolved about the given line.*

45. $y = 1 - \sqrt{x}$, $y = 1$, and $x = 1$; about $y = 1$



46. $y = \sqrt{x} + 2$, $y = 2$, and $x = 1$; about $y = 2$

47. $x = 2 - \sec y$, $x = 2$, $y = \dfrac{\pi}{3}$, and $y = 0$; about $x = 2$



48. $y = 1 - \sqrt{x}$, $x = 1$, and $y = 1$; about $x = 1$

49. $x = 0$, $y = \sqrt{x}$, and $y = 1$; about $y = 1$

50. $x = 0$, $y = \sqrt{x}$, and $y = 2$; about $x = 4$

51. $y = 2\sin x$ and $y = 0$ on $[0, \pi]$; about $y = -2$

52. $y = \dfrac{x^2}{4}$ and the y-axis on the interval $0 \le y \le 1$; about the line $x = -1$

53. $y = \sin x$ and $y = 1 - \sin x$ on the interval $\dfrac{\pi}{6} \le x \le \dfrac{5\pi}{6}$; about $y = -1$

54. $y = x$ and $y = 1 + \dfrac{x}{2}$; about $y = 3$

55. $y = 2 - x$ and $y = 2 - 2x$; about $x = 3$

56. **Comparing volumes** The region R is bounded by the graph of $f(x) = 2x(2 - x)$ and the x-axis. Which is greater, the volume of the solid generated when R is revolved about the line $y = 2$ or the volume of the solid generated when R is revolved about the line $y = 0$? Use integration to justify your answer.

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. A pyramid is a solid of revolution.

b. The volume of a hemisphere can be computed using the disk method.

c. Let R_1 be the region bounded by $y = \cos x$ and the x-axis on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. Let R_2 be the region bounded by $y = \sin x$ and the x-axis on $[0, \pi]$. The volumes of the solids generated when R_1 and R_2 are revolved about the x-axis are equal.

Explorations and Challenges »

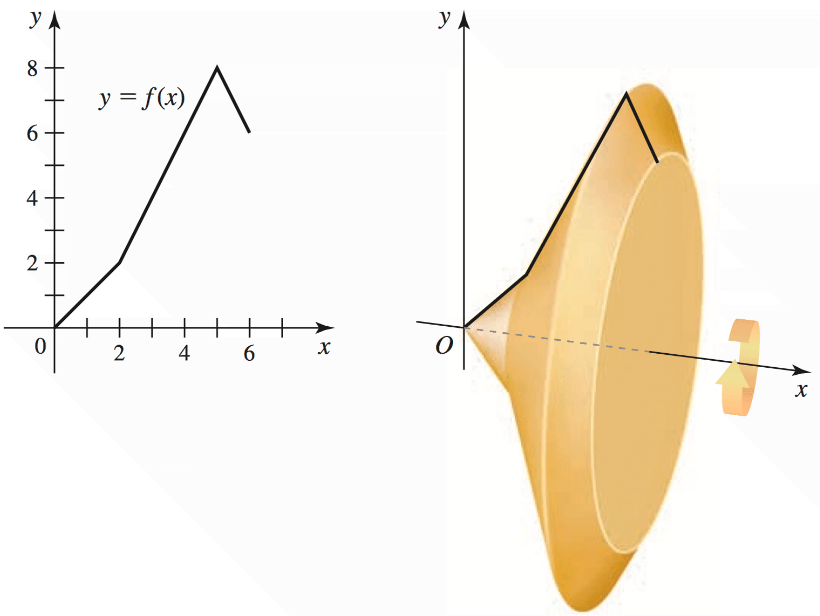
58. Use calculus to find the volume of a tetrahedron (pyramid with four triangular faces), all of whose edges have length 4.

59. **Fermat's volume calculation (1636)** Let R be the region bounded by the curve $y = \sqrt{x+a}$ (with $a > 0$), the y-axis, and the x-axis. Let S be the solid generated by rotating R about the y-axis. Let T be the inscribed cone that has the same circular base as S and height $\sqrt{a}$. Show that $\dfrac{\text{volume }(S)}{\text{volume }(T)} = \dfrac{8}{5}$.

60. **Solid from a piecewise function** Let

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 2 \\ 2x - 2 & \text{if } 2 < x \le 5 \\ -2x + 18 & \text{if } 5 < x \le 6. \end{cases}$$

Find the volume of the solid formed when the region bounded by the graph of f, the x-axis, and the line $x = 6$ is revolved about the x-axis.



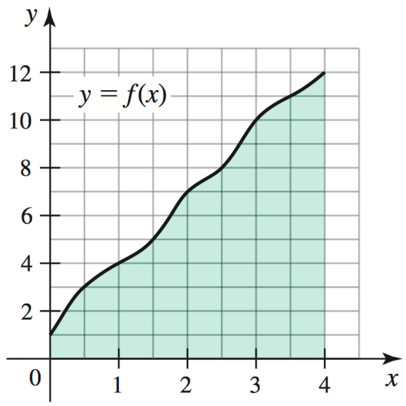
61. **Solids from integrals** Sketch a solid of revolution whose volume by the disk method is given by the following integrals. Indicate the function that generates the solid. Solutions are not unique.

a. $\displaystyle\int_0^{\pi} \pi \sin^2 x \, dx$

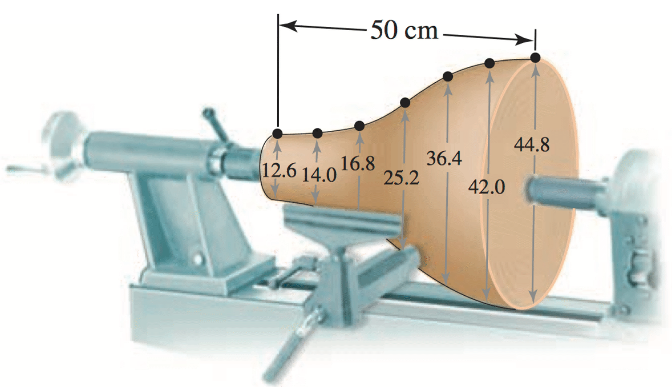
b. $\displaystyle\int_0^{2} \pi \left(x^2 + 2x + 1\right) dx$

62. Volume of a cup A 6-inch-tall plastic cup is shaped like a surface obtained by rotating a line segment in the first quadrant about the x-axis. Given that the radius of the base of the cup is 1 inch, the radius of the top of the cup is 2 inches, and the cup is filled to the brim with water, use integration to approximate the volume of the water in the cup.

63. Estimating volume Suppose the region bounded by the curve $y = f(x)$ from $x = 0$ to $x = 4$ (see figure) is revolved about the x-axis to form a solid of revolution. Use left, right, and midpoint Riemann sums, with $n = 4$ subintervals of equal length, to estimate the volume of the solid of revolution.



T 64. Volume of a wooden object A solid wooden object turned on a lathe has a length of 50 cm and diameters (measured in cm) shown in the figure. (A lathe is a tool that spins and cuts a block of wood so that it has circular cross sections.) Use left Riemann sums with uniformly spaced grid points to estimate the volume of the object.



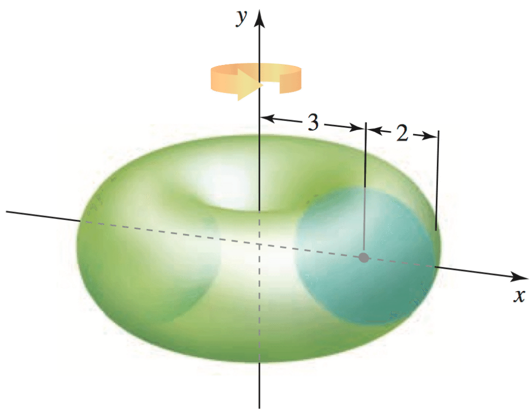
65. Cylinder, cone, hemisphere A right circular cylinder with height R and radius R has a volume of $V_C = \pi R^3$ (height = radius).

a. Find the volume of the cone that is inscribed in the cylinder with the same base as the cylinder and height R. Express the volume in terms of V_C.

b. Find the volume of the hemisphere that is inscribed in the cylinder with the same base as the cylinder. Express the volume in terms of V_C.

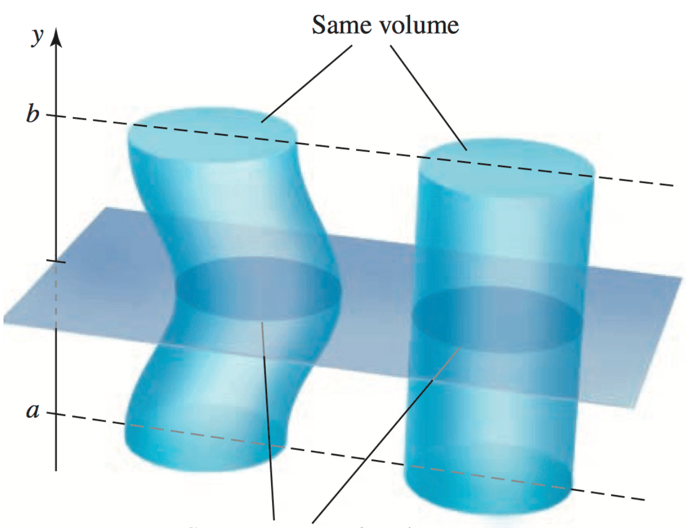
66. Water in a bowl A hemispherical bowl of radius 8 inches is filled to a depth of h inches, where $0 \le h \le 8$. Find the volume of water in the bowl as a function of h. (Check the special cases $h = 0$ and $h = 8$.)

T 67. A torus (doughnut) Find the volume of the torus formed when the circle of radius 2 centered at (3, 0) is revolved about the y-axis. Use geometry to evaluate the integral.



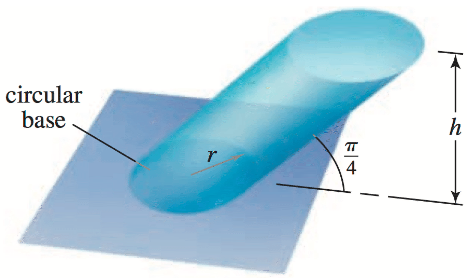
68. Which is greater? Let R be the region bounded by $y = x^2$ and $y = \sqrt{x}$. Use integration to determine which is greater: the volume of the solid generated when R is revolved about the x-axis or about the line $y = 1$.

69. Cavalieri's principle *Cavalieri's principle* states that if two solids with equal altitudes have the same cross-sectional areas at every height, then they have equal volumes (see figure).



a. Use the theory of this section to justify Cavalieri's principle.

b. Use Cavalieri's principle to find the volume of a circular cylinder of radius r and height h whose axis is at an angle of $\frac{\pi}{4}$ to the base.



70. Limiting volume Consider the region R in the first quadrant bounded by $y = x^{1/n}$ and $y = x^n$, where $n > 1$ is a positive number.

a. Find the volume $V(n)$ of the solid generated when R is revolved about the x-axis. Express your answer in terms of n.

b. Evaluate $\lim_{n \to \infty} V(n)$. Interpret this limit geometrically.