6.2 Regions Between Curves

In this section, the method for finding the area of a region bounded by a single curve is generalized to regions bounded by two or more curves. Consider two functions f and g that are continuous on an interval $[a, b]$ on which $f(x) \geq g(x)$ (**Figure 6.11**). The goal is to find the area A of the region bounded by the two curves and the vertical lines $x = a$ and $x = b$.

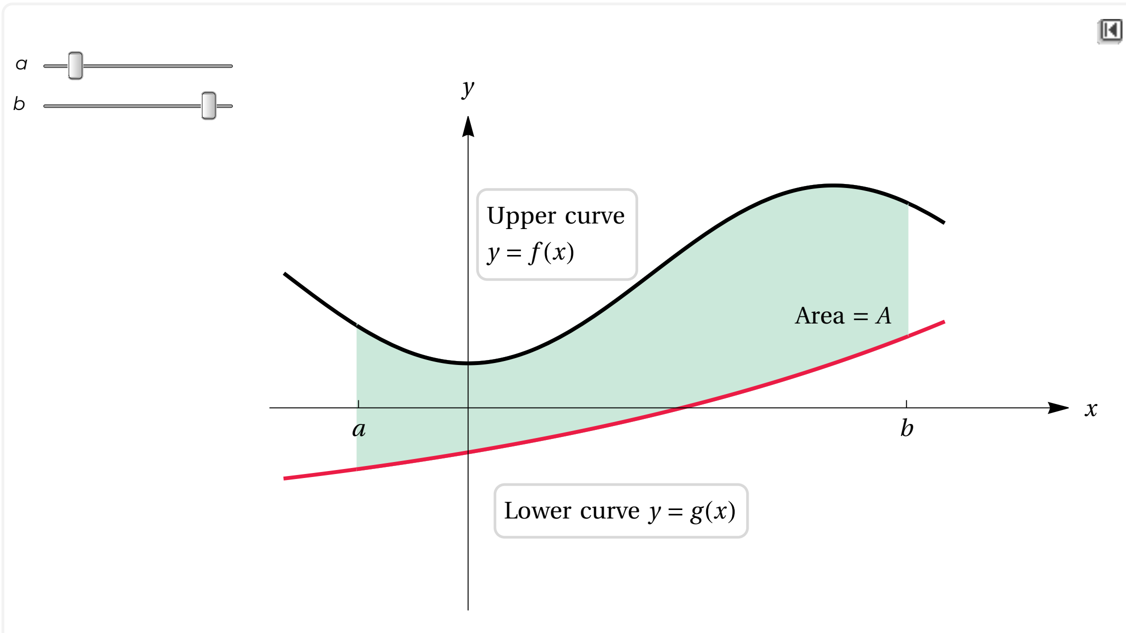


Figure 6.11

Integrating with Respect to *x* »

Once again we rely on the *slice-and-sum* strategy (Section 5.2) for finding areas by Riemann sums. The interval $[a, b]$ is partitioned into n subintervals using uniformly spaced grid points separated by a distance $\Delta x = \dfrac{b-a}{n}$ (**Figure 6.12**).

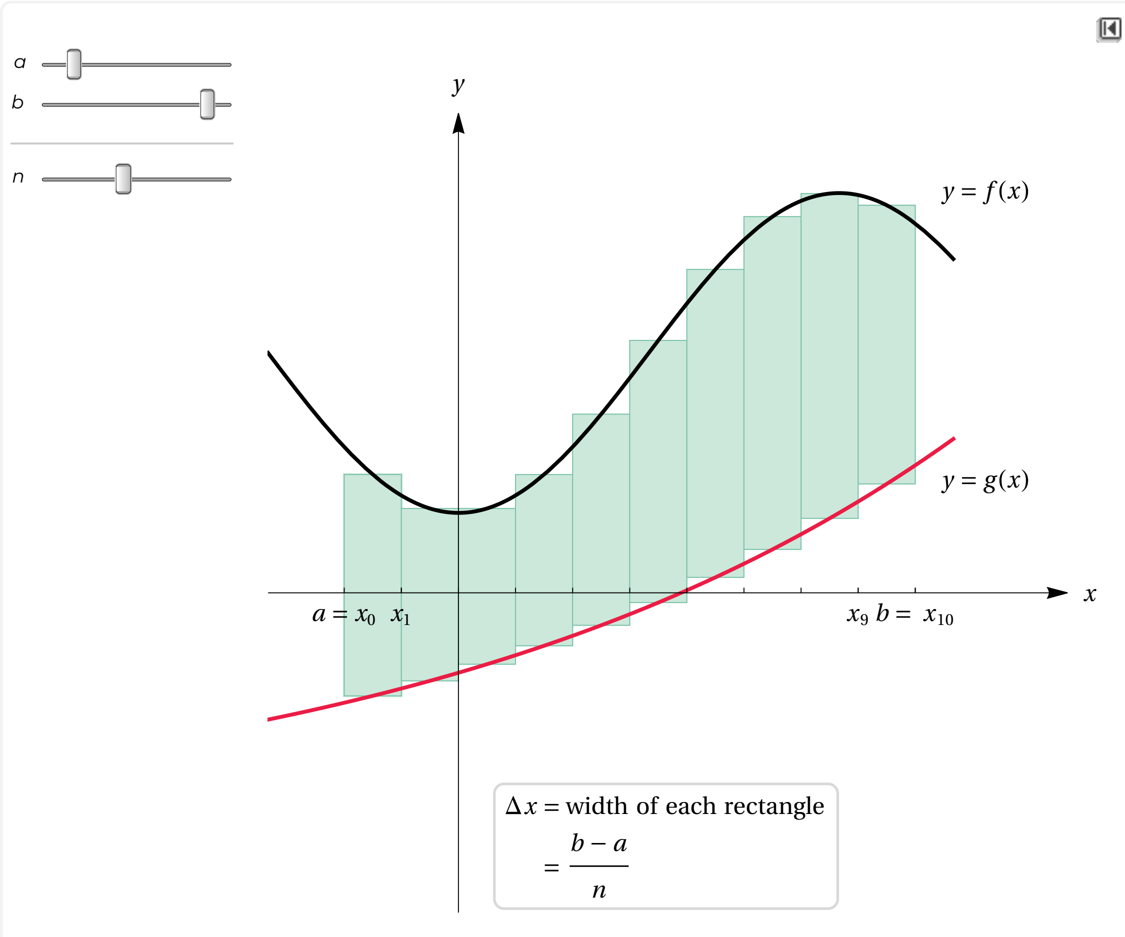


Figure 6.12

On each subinterval, we build a rectangle extending from the lower curve to the upper curve. On the kth subinterval, a point x_k^* is chosen, and the height of the corresponding rectangle is taken to be $f(x_k^*) - g(x_k^*)$. Therefore, the area of the kth rectangle is $(f(x_k^*) - g(x_k^*))\,\Delta x$ (**Figure 6.13**). Summing the areas of the n rectangles gives an approximation to the area of the region between the curves:

$$A \approx \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\,\Delta x.$$

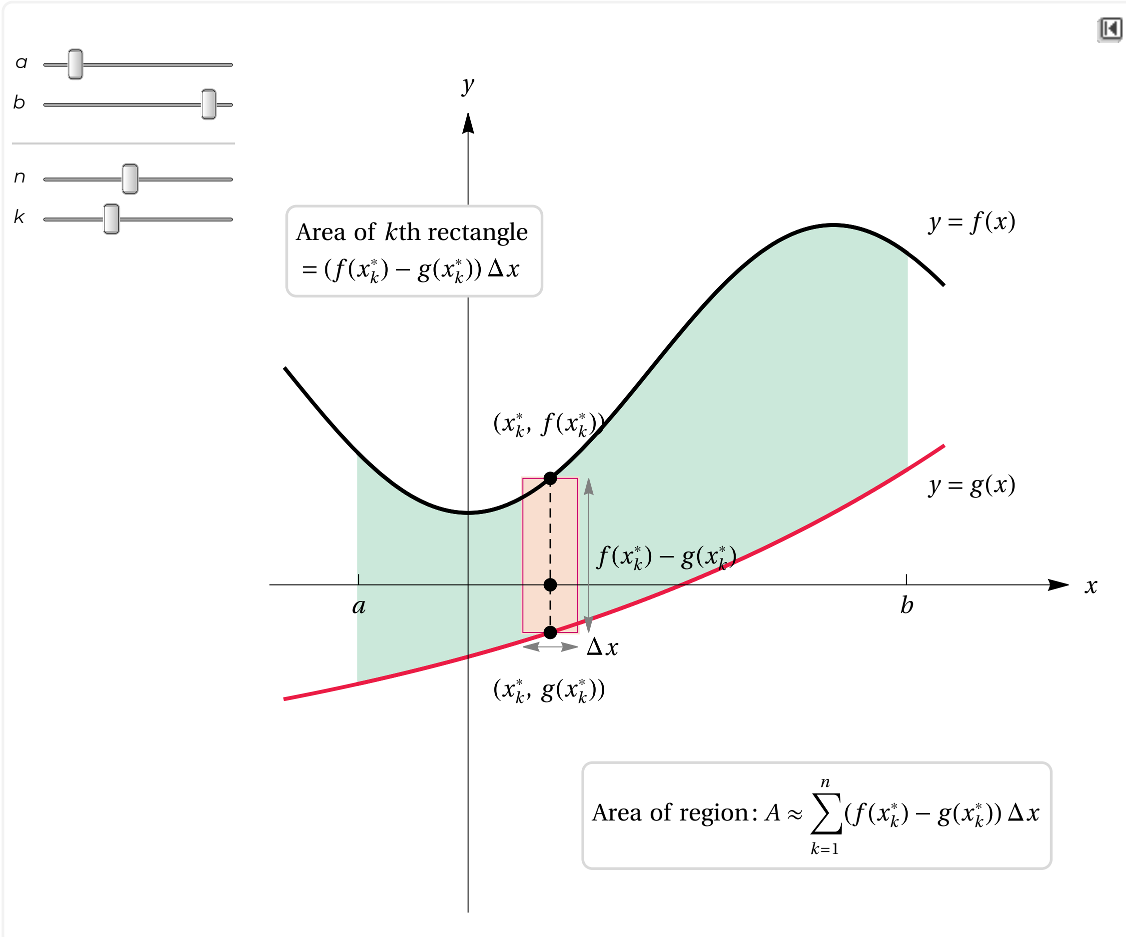


Figure 6.13

As the number of grid points increases, Δx approaches zero and these sums approach the area between the curves; that is,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\,\Delta x.$$

The limit of these Riemann sums is a definite integral of the function $f - g$.

DEFINITION **Area of a Region Between Two Curves**

Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))\,dx.$$

Note »

It is helpful to interpret the area formula: $f(x) - g(x)$ is the length of a rectangle and dx represents its width. We sum (integrate) the areas of the rectangles $(f(x) - g(x))\,dx$ to obtain the area of the region.

Quick Check 1 In the area formula for a region between two curves, verify that if the lower curve is $g(x) = 0$, the formula becomes the usual formula for the area of the region bounded by $y = f(x)$ and the x-axis. ◆

Answer »

If $g(x) = 0$ and $f(x) \geq 0$, then the area between the curves is $\int_a^b (f(x) - 0)\, dx = \int_a^b f(x)\, dx$, which is the area between $y = f(x)$ and the x-axis.

EXAMPLE 1 Area between curves

Find the area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$ (**Figure 6.14**).

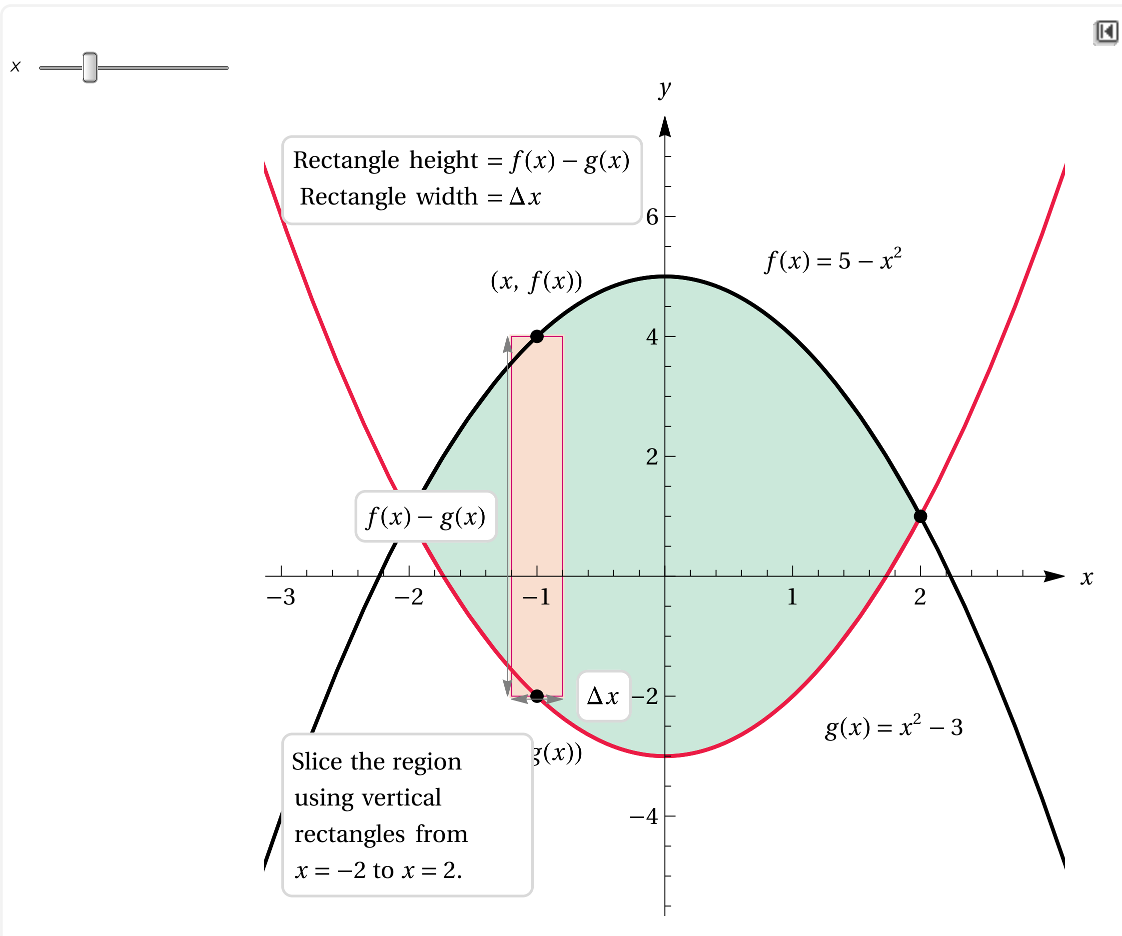


Figure 6.14

SOLUTION »

A key step in the solution of many area problems is finding the intersection points of the boundary curves, which often determine the limits of integration. The intersection points of these two curves satisfy the equation $5 - x^2 = x^2 - 3$. The solutions to this equation are $x = -2$ and $x = 2$, which become the lower and upper limits of integration, respectively. The graph of f is the upper curve and the graph of g is the lower curve on the interval $[-2, 2]$. Therefore, the area of the region is

$$\begin{aligned} A &= \int_{-2}^{2}\left(\left(5-x^2\right)-\left(x^2-3\right)\right)dx && \text{Substitute for } f \text{ and } g. \\ &= 2\int_{0}^{2}\left(8-2x^2\right)dx && \text{Simplify and use symmetry.} \\ &= 2\left(8x-\frac{2}{3}x^3\right)\Bigg|_0^2 && \text{Fundamental Theorem} \\ &= \frac{64}{3}. && \text{Simplify.} \end{aligned}$$

Notice how the symmetry of the problem simplifies the integration. Also note that the area formula $A=\int_a^b (f(x)-g(x))\,dx$ is valid even if one or both curves lie below the x-axis. However, you must be sure that $f(x)\ge g(x)$ on $[a, b]$.

Related Exercises 9–10 ◆

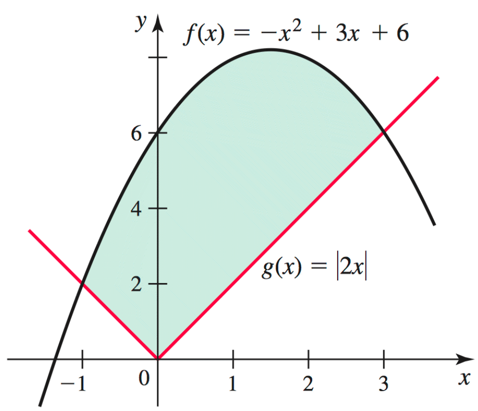
Quick Check 2 Interpret the area formula in the form $A=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$, where $f(x)\ge g(x)\ge 0$ on $[a, b]$. ◆

Answer »

$\int_a^b f(x)\,dx$ is the area of the region between the graph of f and the x-axis. $\int_a^b g(x)\,dx$ is the area of the region between the graph of g and the x-axis. The difference of the two integrals is the area of the region between the graphs of f and g.

EXAMPLE 2 Compound region

Find the area of the region bounded by the graphs of $f(x)=-x^2+3x+6$ and $g(x)=|2x|$ (**Figure 6.15a**).



(a)

Figure 6.15a

SOLUTION »

The lower boundary of the region in question is bounded by two different branches of the absolute value function. In situations like this, the region is divided into two (or more) subregions, whose areas are found independently and then summed; these regions are labeled R_1 and R_2 (**Figure 6.15b**). By the definition of absolute value,

$$g(x) = |2\,x| = \begin{cases} 2\,x & \text{if } x \geq 0 \\ -2\,x & \text{if } x < 0. \end{cases}$$

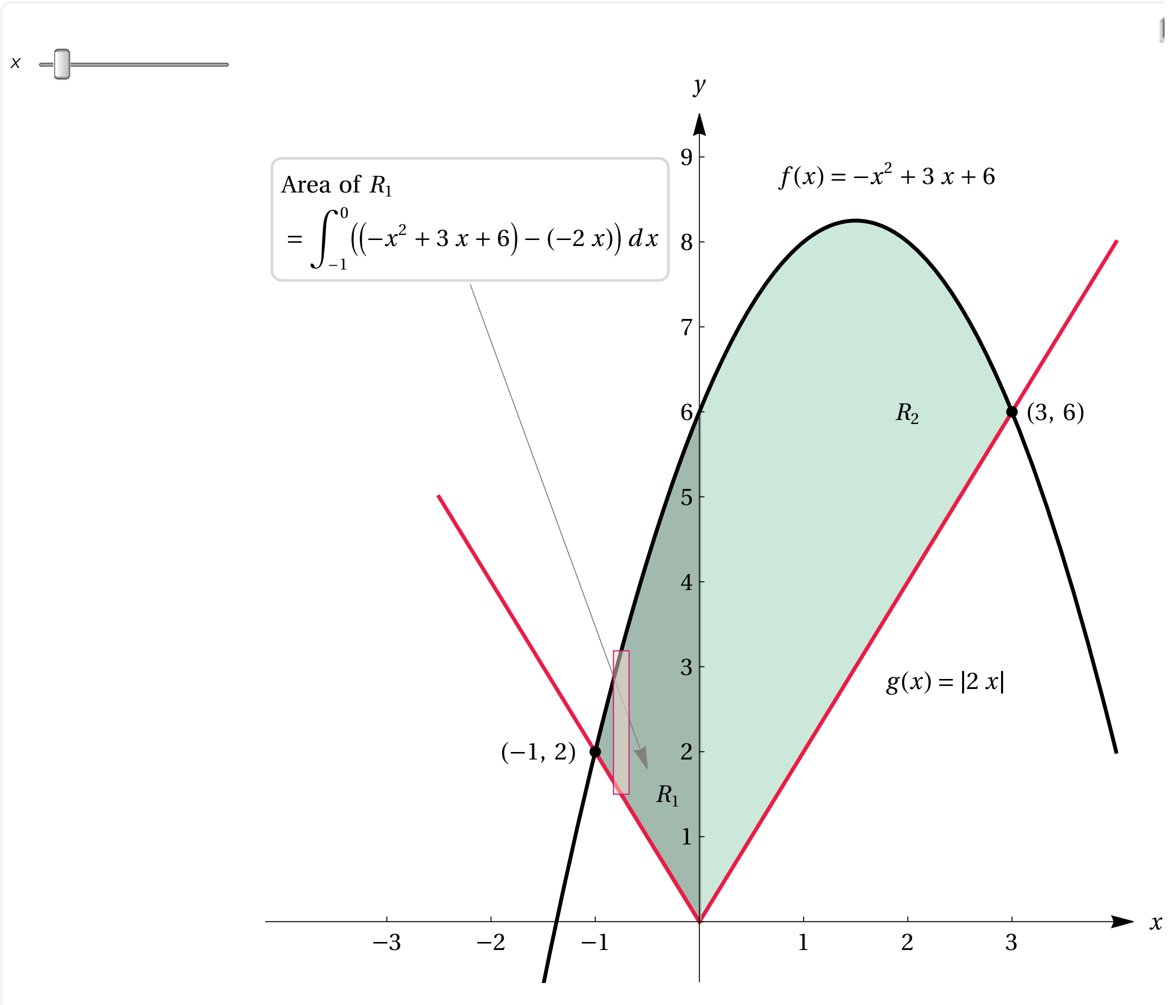


Figure 6.15 b

The left intersection point of f and g satisfies $-2\,x = -x^2 + 3\,x + 6$, or $x^2 - 5\,x - 6 = 0$. Solving for x, we find that $(x + 1)\,(x - 6) = 0$, which implies $x = -1$ or $x = 6$; only the first solution is relevant. The right intersection point of f and g satisfies $2\,x = -x^2 + 3\,x + 6$; you should verify that the relevant solution in this case is $x = 3$.

Note »

Given these points of intersection, we see that the region R_1 is bounded by $y = -x^2 + 3\,x + 6$ and $y = -2\,x$ on the interval $[-1, 0]$. Similarly, region R_2 is bounded by $y = -x^2 + 3\,x + 6$ and $y = 2\,x$ on $[0, 3]$ (Figure 6.15b). Therefore,

$$A = \underbrace{\int_{-1}^{0}\left(\left(-x^2+3x+6\right)-(-2x)\right]dx}_{\text{area of region } R_1} + \underbrace{\int_{0}^{3}\left(\left(-x^2+3x+6\right)-2x\right)dx}_{\text{area of region } R_2}$$

$$= \int_{-1}^{0}\left(-x^2+5x+6\right)dx + \int_{0}^{3}\left(-x^2+x+6\right)dx \qquad \text{Simplify.}$$

$$= \left(-\frac{x^3}{3}+\frac{5x^2}{2}+6x\right)\Bigg|_{-1}^{0} + \left(-\frac{x^3}{3}+\frac{x^2}{2}+6x\right)\Bigg|_{0}^{3} \qquad \text{Fundamental Theorem}$$

$$= 0-\left(\frac{1}{3}+\frac{5}{2}-6\right)+\left(-9+\frac{9}{2}+18\right)-0=\frac{50}{3}. \qquad \text{Simplify.}$$

Related Exercises 15–16 ◆

Integrating with Respect to *y* »

There are occasions when it is convenient to reverse the roles of x and y. Consider the regions shown in **Figure 6.16** that are bounded by the graphs of $x = f(y)$ and $x = g(y)$, where $f(y) \ge g(y)$, for $c \le y \le d$ (the graph of f lies to the right of the graph of g). The lower and upper boundaries of the regions are $y = c$ and $y = d$, respectively.

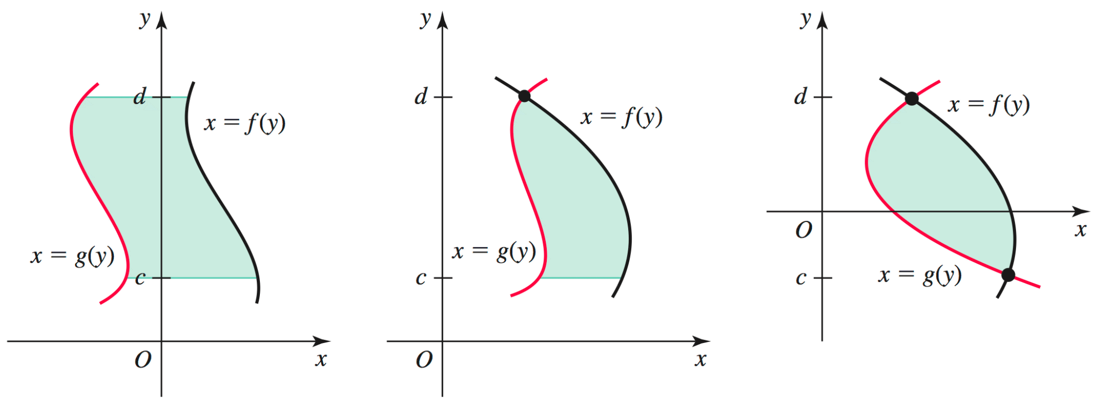


Figure 6.16

In cases such as these, we treat y as the independent variable and divide the interval $[c, d]$ into n subintervals of width $\Delta y = \dfrac{d-c}{n}$ (**Figure 6.17**).

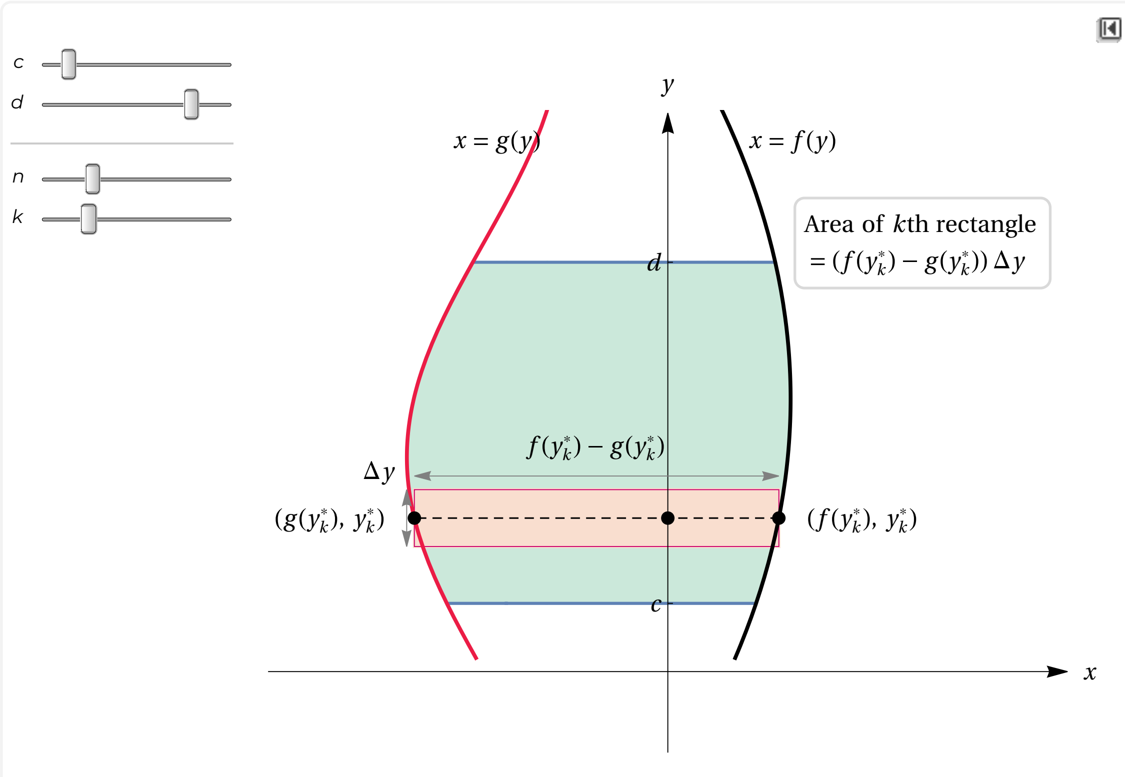


Figure 6.17

On the kth subinterval, a point y_k^* is selected and we construct a rectangle that extends from the left curve to the right curve. The kth rectangle has length $f(y_k^*) - g(y_k^*)$, so the area of the kth rectangle is $(f(y_k^*) - g(y_k^*))\,\Delta y$. The area of the region is approximated by the sum of the areas of the rectangles. In the limit as $n \to \infty$ and $\Delta y \to 0$, the area of the region is given as the definite integral

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} (f(y_k^*) - g(y_k^*))\,\Delta y = \int_c^d (f(y) - g(y))\,dy.$$

DEFINITION **Area of a Region Between Two Curves with Respect to y**

Suppose that f and g are continuous functions with $f(y) \ge g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\,dy.$$

Note »

This area formula is identical to the one given in an earlier Definition box (**Area of a Region Between Two Curves**); it is now expressed with respect to the y-axis. In this case, $f(y) - g(y)$ is the length of a rectangle and dy represents its width. We sum (integrate) the areas of the rectangles $(f(y) - g(y))\,dy$ to obtain the area of the region.

EXAMPLE 3 Integrating with respect to y

Find the area of the region R bounded by the graphs of $y = x^3$, $y = x + 6$, and the x-axis.

SOLUTION »

The area of this region could be found by integrating with respect to x. But this approach requires splitting the region into two pieces (Figure 6.18).

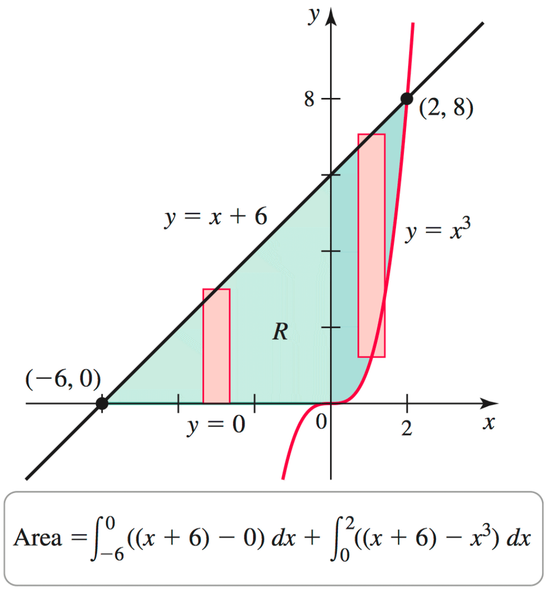


Figure 6.18

Alternatively, we can view y as the independent variable, express the bounding curves as functions of y, and make horizontal slices parallel to the x-axis (Figure 6.19).

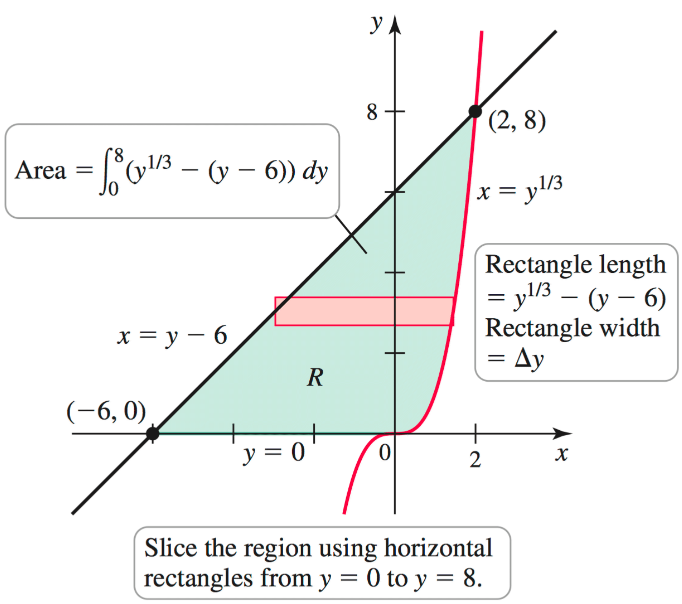


Figure 6.19

Solving for x in terms of y, the right curve $y = x^3$ becomes $x = f(y) = y^{1/3}$. The left curve $y = x + 6$ becomes $x = g(y) = y - 6$. The intersection point of the curves satisfies the equation $y^{1/3} = y - 6$, or $y = (y-6)^3$. Expanding this equation gives the cubic equation

$$y^3 - 18y^2 + 107y - 216 = (y-8)\left(y^2 - 10y + 27\right) = 0,$$

whose only real root is $y = 8$. As shown in Figure 6.19, the areas of the slices through the region are summed from $y = 0$ to $y = 8$. Therefore, the area of the region is given by

$$\int_0^8 \left(y^{1/3} - (y-6)\right) dy = \left(\frac{3}{4}y^{4/3} - \frac{y^2}{2} + 6y\right)\Bigg|_0^8 \qquad \text{Fundamental Theorem}$$

$$= \left(\frac{3}{4}\cdot 16 - 32 + 48\right) - 0 = 28. \qquad \text{Simplify.}$$

Note »

You may use synthetic division or a root finder to factor the cubic polynomial in Example 3. Then the quadratic formula shows that the equation

$$y^2 - 10y + 27 = 0$$

has no real roots.

Related Exercises 19–20 ◆

Quick Check 3 The region R is bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$, and the x-axis. Express the area of R in terms of (a) integral(s) with respect to x and (b) integral(s) with respect to y. ◆

Answer »

a. $\int_0^2 \sqrt{x}\,dx + \int_2^4 \left(\sqrt{x} - x + 2\right) dx$

b. $\int_0^2 \left(y + 2 - y^2\right) dy.$

EXAMPLE 4 Calculus and geometry

Find the area of the region R in the first quadrant bounded by the curves $y = x^{2/3}$ and $y = x - 4$ (**Figure 6.20**).

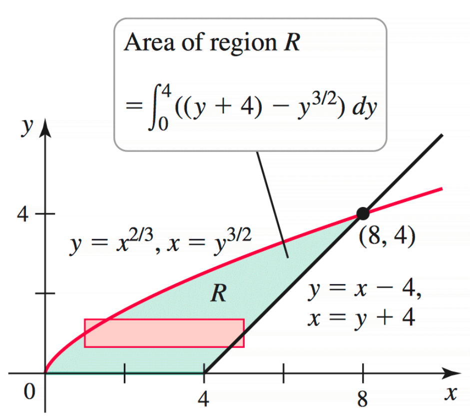


Figure 6.20

SOLUTION »

Slicing the region vertically and integrating with respect to x requires two integrals. Slicing the region horizontally requires a single integral with respect to y. The second approach appears to involve less work.

Slicing horizontally, the right bounding curve is $x = y + 4$ and the left bounding curve is $x = y^{3/2}$. The two curves intersect at (8, 4), so the limits of integration are $y = 0$ and $y = 4$. The area of R is

$$\int_0^4 \left(\underbrace{(y+4)}_{\text{right curve}} - \underbrace{y^{3/2}}_{\text{left curve}} \right) dy = \left(\frac{y^2}{2} + 4y - \frac{2}{5} y^{5/2} \right) \Bigg|_0^4 = \frac{56}{5}.$$

Can this area be found using a different approach? Sometimes it helps to use geometry. Notice that the region R can be formed by taking the entire region under the curve $y = x^{2/3}$ on the interval [0, 8] and then removing a triangle whose base is the interval [4, 8] (**Figure 6.21**). The area of the region R_1 under the curve $y = x^{2/3}$ is

$$\int_0^8 x^{2/3}\, dx = \frac{3}{5} x^{5/3} \Bigg|_0^8 = \frac{96}{5}.$$

The triangle R_2 has a base of length 4 and a height of 4, so its area is $\frac{1}{2} \cdot 4 \cdot 4 = 8$. Therefore, the area of R is $\frac{96}{5} - 8 = \frac{56}{5}$, which agrees with the first calculation.

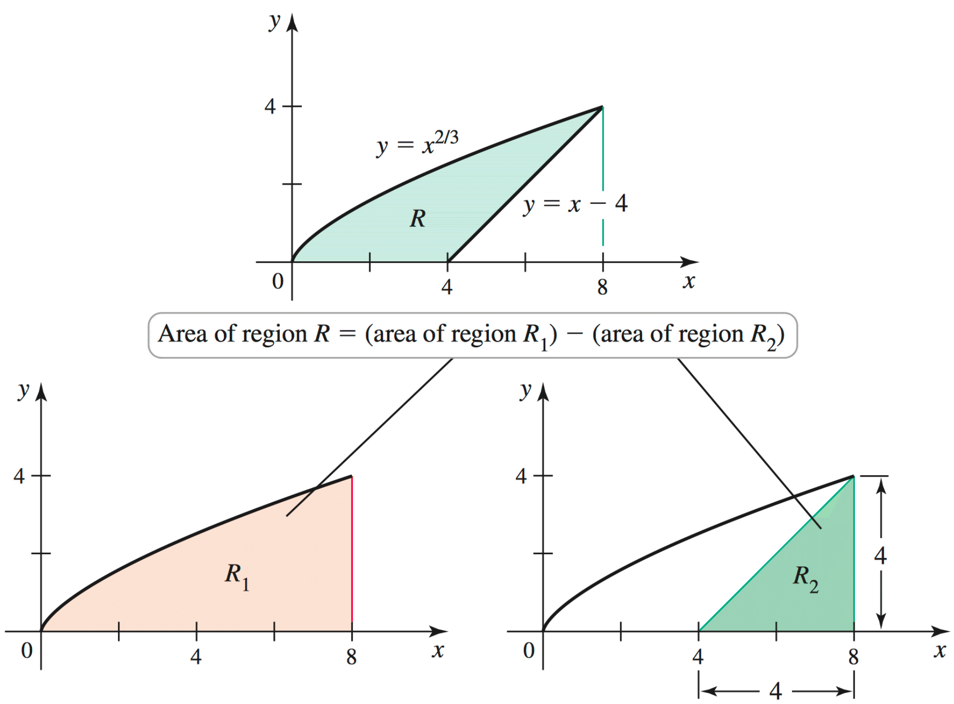


Figure 6.21

Related Exercises 34–36 ◆

Quick Check 4 An alternative way to determine the area of the region in Example 3 (Figure 6.18) is to compute $18 + \int_0^2 (x + 6 - x^3)\,dx$. Why? ◆

Answer »

> The area of the triangle to the left of the y-axis is 18. The area of the region to the right of the y-axis is given by the integral.

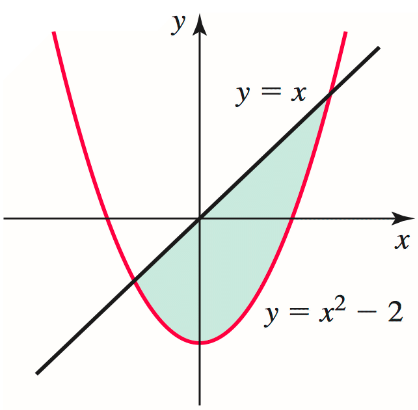
Exercises »

Getting Started »

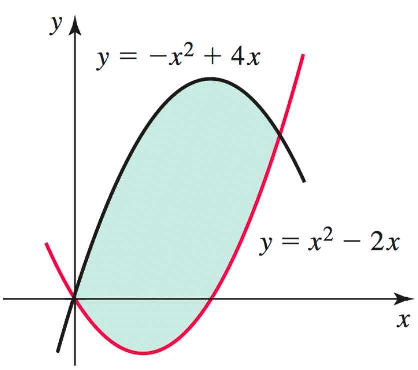
Practice Exercises »

9–30. Finding area *Determine the area of the shaded region in the following figures.*

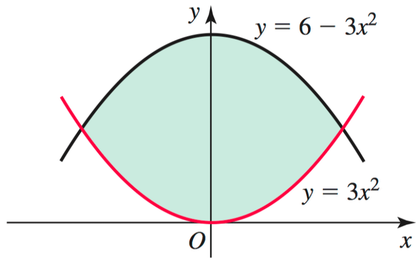
9.



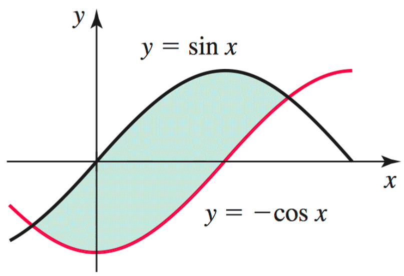
10.



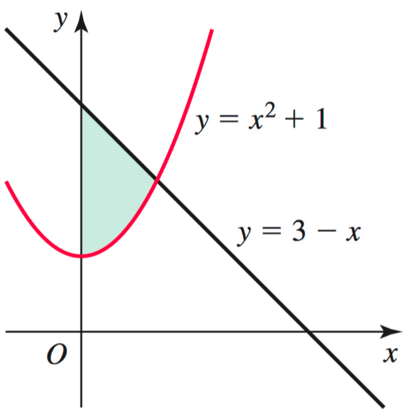
11.



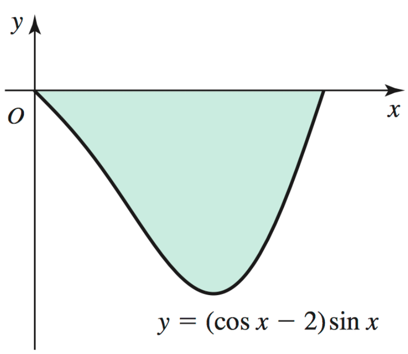
12.



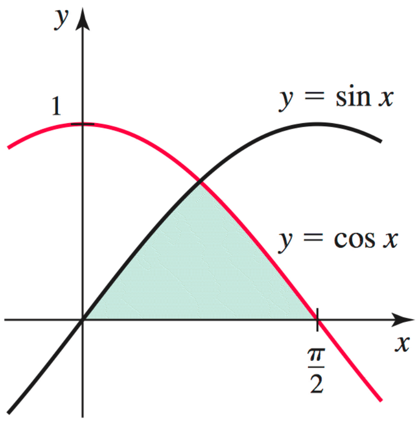
13. (*Hint:* Find the intersection point by inspection.)



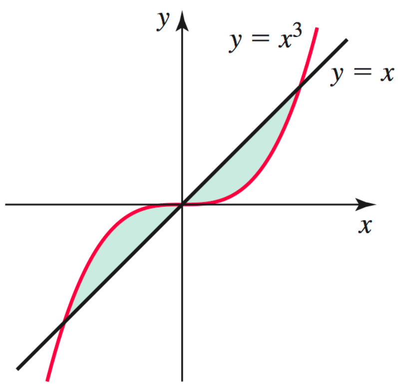
14.



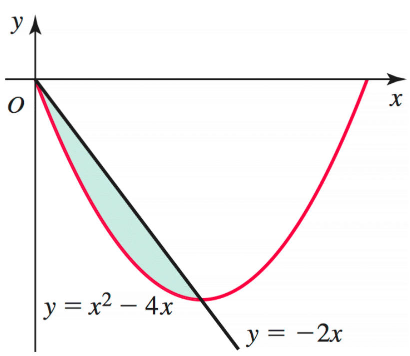
15.



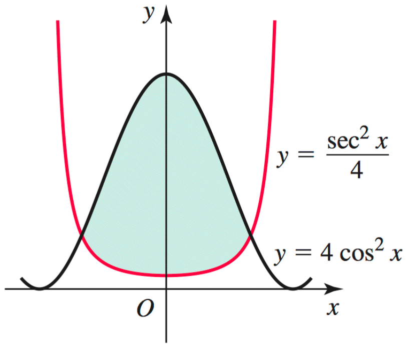
16.



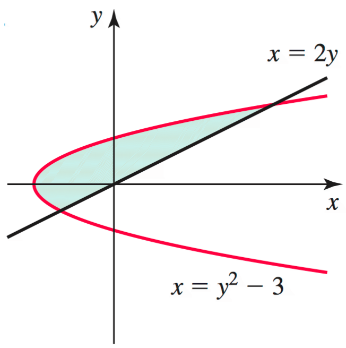
17.



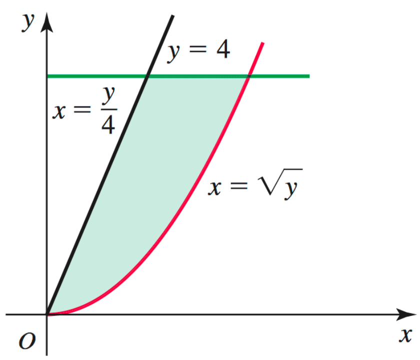
18.


y
$y = \frac{\sec^2 x}{4}$
$y = 4\cos^2 x$
O
x

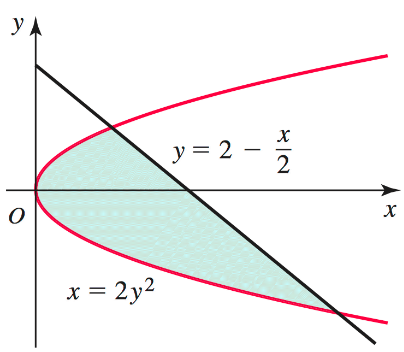
19.


y
$x = 2y$
x
$x = y^2 - 3$

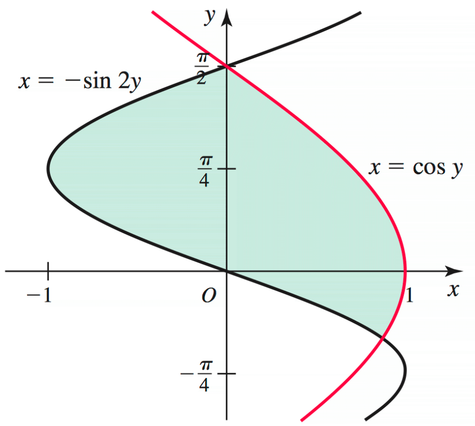
20.


y
$y = 4$
$x = \frac{y}{4}$
$x = \sqrt{y}$
O
x

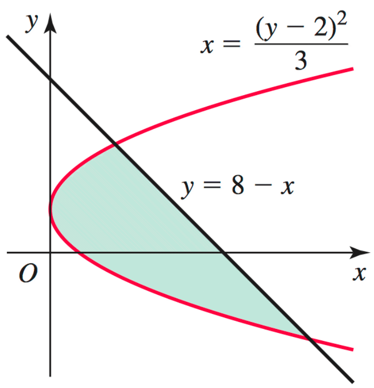
21.


y
$y = 2 - \frac{x}{2}$
O
x
$x = 2y^2$

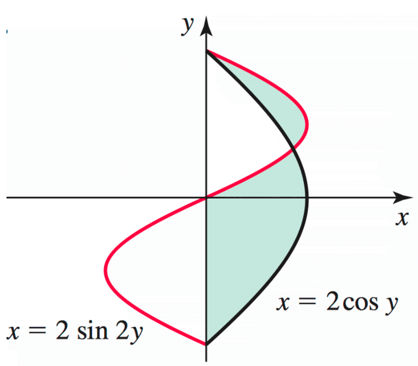
22.


y
$\frac{\pi}{2}$
$x = -\sin 2y$
$\frac{\pi}{4}$
$x = \cos y$
−1
O
1
x
$-\frac{\pi}{4}$

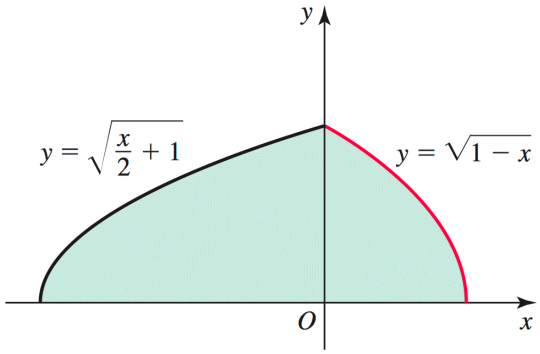
23.


y
$x = \frac{(y-2)^2}{3}$
$y = 8 - x$
O
x

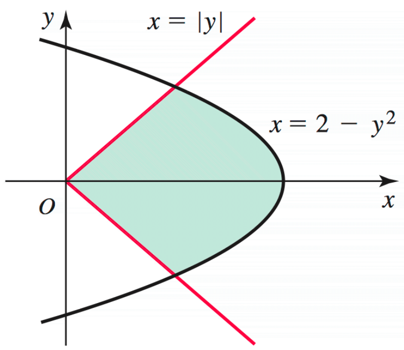
24.


y
x
$x = 2\cos y$
$x = 2\sin 2y$

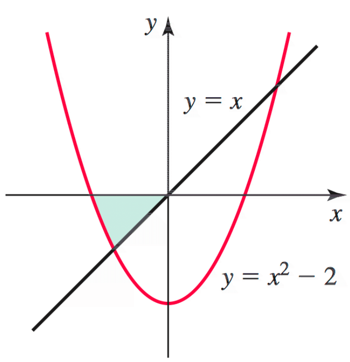
25.



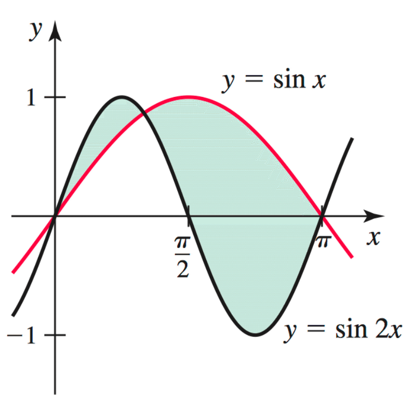
26. The region bounded between $x = 2 - y^2$ and $x = |y|$



27.



28.



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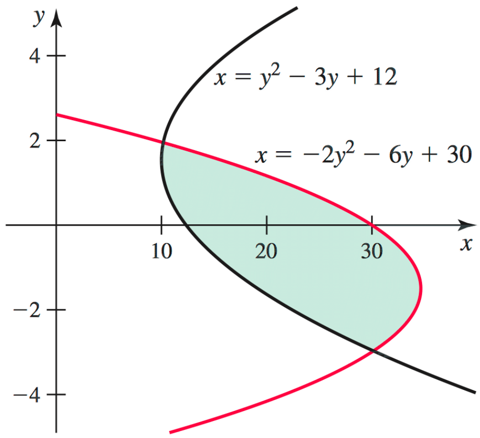
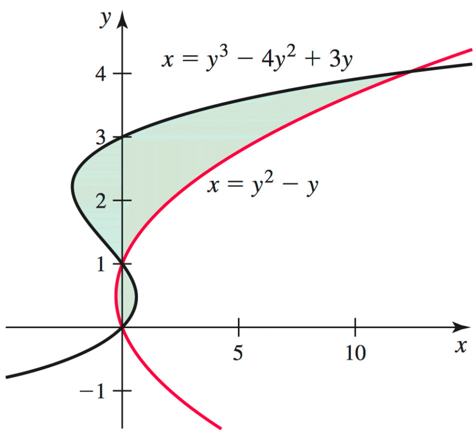
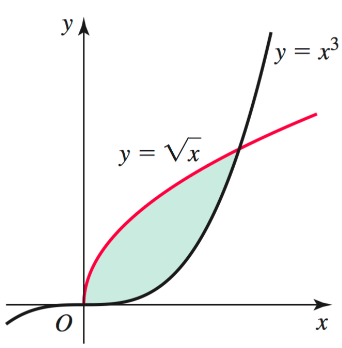


30.

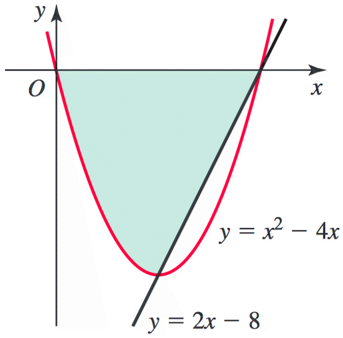


31–32. Two approaches *Express the area of the following shaded regions in terms of (a) one or more integrals with respect to x and (b) one or more integrals with respect to y. You do not need to evaluate the integrals.*

31.



32.



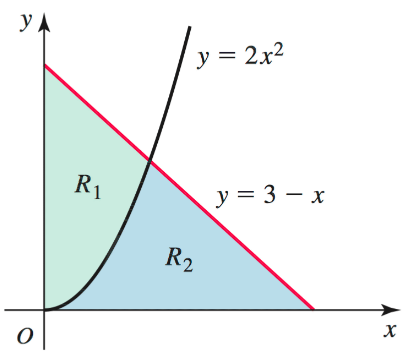
33. Area between velocity curves Two runners, starting at the same location, run along a straight road for 1 hour. The velocity of one runner is $v_1(t) = 7\,t$ and the velocity of the other runner is $v_2(t) = 10\sqrt{t}$. Assume t is measured in hours and the velocities $v_1(t)$ and $v_2(t)$ are measured in km/hr. Determine the area between the curves $y = v_1(t)$ and $y = v_2(t)$, for $0 \le t \le 1$. Interpret the physical meaning of this area.

34–36. Calculus and geometry *For the given regions R_1 and R_2, complete the following steps.*

a. *Find the area of region R_1.*

b. *Find the area of region R_2 using geometry and the answer to part (a).*

34. R_1 is the region in the first quadrant bounded by the y-axis and the curves $y = 2\,x^2$ and $y = 3 - x$; R_2 is the region in the first quadrant bounded by the x-axis and the curves $y = 2\,x^2$ and $y = 3 - x$ (see figure).



T **35.** R_1 is the region in the first quadrant bounded by the line $x = 1$ and the curve $y = 6\,x\left(2 - x^2\right)^2$; R_2 is the region in the first quadrant bounded the curve $y = 6\,x\left(2 - x^2\right)^2$ and the line $y = 6\,x$.

36. R_1 is the region in the first quadrant bounded by the coordinate axes and the curve $x = \cos y$; R_2 is the region bounded by the lines $y = \dfrac{\pi}{2}$ and $x = 1$, and the curve $x = \cos y$.

37–62. Regions between curves *Find the area of the region described in the following exercises.*

37. The region bounded by $y = 4\,x + 4$, $y = 6\,x + 6$, and $x = 4$

38. The region bounded by $y = \cos x$ and $y = \sin x$ on the interval $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

39. The region bounded by $y = 2x^2$ and $y = x^2 + 4$

40. The region bounded by $y = 6x$ and $y = 3x^2 - 6x$

41. The region bounded by $y = 2x$ and $y = 2x^2 - 4$

42. The region bounded by $y = 24\sqrt{x}$ and $y = 3x^2$

43. The region bounded by $y = x$, $y = \frac{1}{x^2}$, $y = 0$, and $x = 2$

44. The region in the first quadrant on the interval $[0, 2]$ bounded by $y = 4x - x^2$ and $y = 4x - 4$

45. The region bounded by $y = 2 - |x|$ and $y = x^2$

46. The region bounded by $y = x^3$ and $y = 9x$

47. The region bounded by $y = |x - 3|$ and $y = \frac{x}{2}$

48. The region bounded by $y = 3x - x^3$ and $y = -x$

49. The region in the first quadrant bounded by $y = x^{2/3}$ and $y = 4$

50. The region in the first quadrant bounded by $y = 2$ and $y = 2\sin x$ on the interval $\left[0, \frac{\pi}{2}\right]$

51. The region bounded by $y = \frac{x}{4}$, and $x = y^3$

T **52.** The region below the line $y = 2$ and above the curve $y = \sec^2 x$ on the interval $\left[0, \frac{\pi}{4}\right]$

T **53.** The region between the line $y = x$ and the curve $y = 2x\sqrt{1 - x^2}$ in the first quadrant

54. The region bounded by $x = y^2 - 4$ and $y = \frac{x}{3}$

55. The region bounded by $y = \sqrt{x}$, $y = 2x - 15$, and $y = 0$

56. The region bounded by $y = 2$ and $y = x^{2/3} + 1$

57. The region bounded by $y = x^2 - 2x + 1$ and $y = 5x - 9$

58. The region bounded by $x = y(y - 1)$ and $x = -y(y - 1)$

59. The region bounded by $x = y(y - 1)$ and $y = \frac{x}{3}$

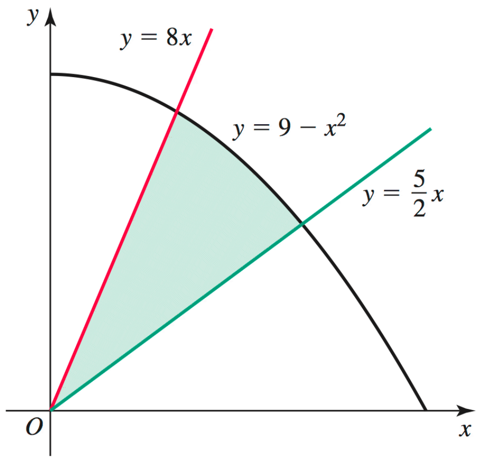
60. The region bounded by $y = \sin x$ and $y = x(x - \pi)$ on the interval $[0, \pi]$

61. The region in the first quadrant bounded by $y = 1 - \cos x$ and $y = \sin x$ on the interval $[\pi/2, 2\pi]$

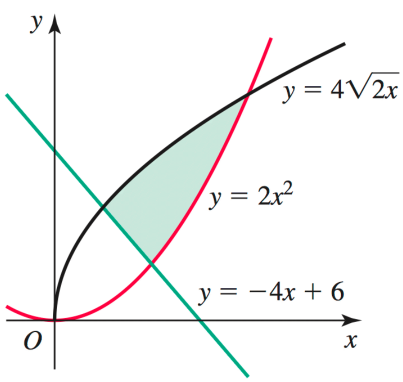
62. The region in the first quadrant bounded by $y = x^{-2}$, $y = 8x$, and $y = \frac{x}{8}$

63–64. Complicated regions *Find the area of the regions shown in the following figures.*

63.



64.



65. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The area of the region bounded by $y = x$ and $x = y^2$ can be found only by integrating with respect to x.

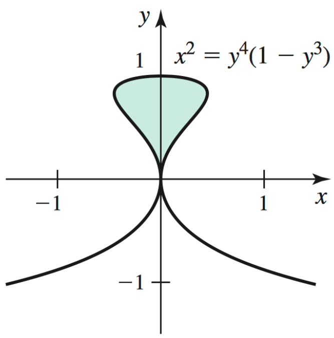
b. The area of the region between $y = \sin x$ and $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$ is $\int_0^{\pi/2} (\cos x - \sin x)\, dx$.

c. $\int_0^1 \left(x - x^2\right) dx = \int_0^1 \left(\sqrt{y} - y\right) dy$.

Explorations and Challenges »

66. Differences of even functions Assume f and g are even, integrable functions on $[-a, a]$, where $a > 1$. Suppose $f(x) > g(x) > 0$ on $[-a, a]$ and that the area bounded by the graphs of f and g on $[-a, a]$ is 10. What is the value of $\int_0^{\sqrt{a}} x\left(f\left(x^2\right) - g\left(x^2\right)\right) dx$?

67. Area of a curve defined implicitly Determine the area of the shaded region bounded by the curve $x^2 = y^4\left(1 - y^3\right)$ (see figure).



68–71. Roots and powers *Find the area of the following regions, expressing your results in terms of the positive integer $n \geq 2$.*

68. The region bounded by $f(x) = x$ and $g(x) = x^n$, for $x \geq 0$

69. The region bounded by $f(x) = x$ and $g(x) = x^{1/n}$, for $x \geq 0$

70. The region bounded by $f(x) = x^{1/n}$ and $g(x) = x^n$, for $x \geq 0$

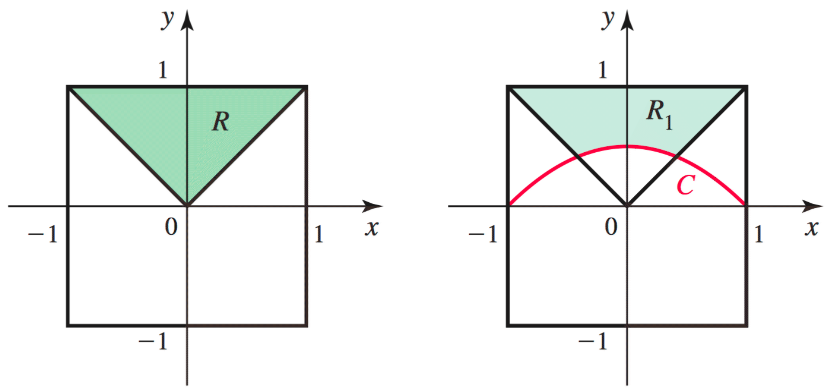
71. Let A_n be the area of the region bounded by $f(x) = x^{1/n}$ and $g(x) = x^n$ on the interval $[0, 1]$, where n is a positive integer. Evaluate $\lim_{n\to\infty} A_n$ and interpret the result.

72–73. Bisecting regions *For each region R, find the horizontal line $y = k$ that divides R into two subregions of equal area.*

72. R is the region bounded by $y = 1 - x$, the x-axis, and the y-axis.

73. R is the region bounded by $y = 1 - |x - 1|$ and the x-axis.

74. Geometric probability Suppose a dartboard occupies the square $\{(x, y): 0 \leq |x| \leq 1,\ 0 \leq |y| \leq 1\}$. A dart is thrown randomly at the board many times (meaning it is equally likely to land at any point in the square). What fraction of the dart throws land closer to the edge of the board than to the center? Equivalently, what is the probability that the dart lands closer to the edge of the board than to the center? Proceed as follows.

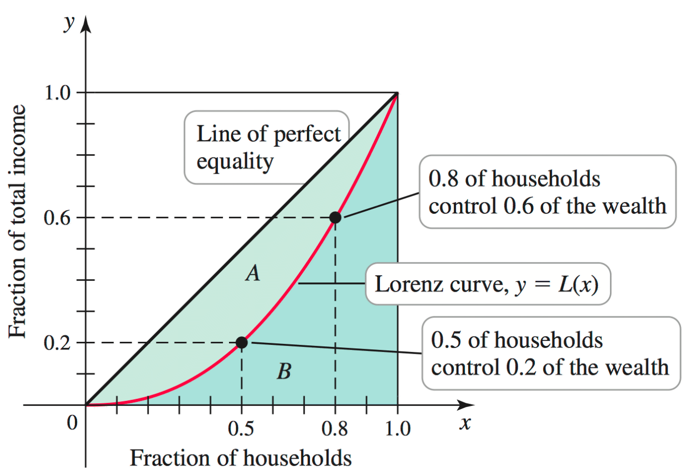


a. Argue that by symmetry it is necessary to consider only one quarter of the board, say the region $R: \{(x, y) : |x| \le y \le 1\}$.

b. Find the curve C in this region that is equidistant from the center of the board and the top edge of the board (see figure).

c. The probability that the dart lands closer to the edge of the board than the center is the ratio of the area of the region R_1 above C to the area of the entire region R. Compute this probability.

T **75. Lorenz curves and the Gini index** A **Lorenz curve** is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth. (See Guided Project *Distribution of Wealth* for more on Lorenz curves.)



a. A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the **line of perfect equality**. Explain why this line is given this name.

b. Explain why a Lorenz curve satisfies the conditions $L(0) = 0$, $L(1) = 1$, $L(x) \le x$, and $L'(x) \ge 0$ on $[0, 1]$.

c. Graph the Lorenz curves $L(x) = x^p$ corresponding to $p = 1.1, 1.5, 2, 3$, and 4. Which value of p corresponds to the *most* equitable distribution of wealth (closest to the line of perfect equality)? Which value of p corresponds to the *least* equitable distribution of wealth? Explain.

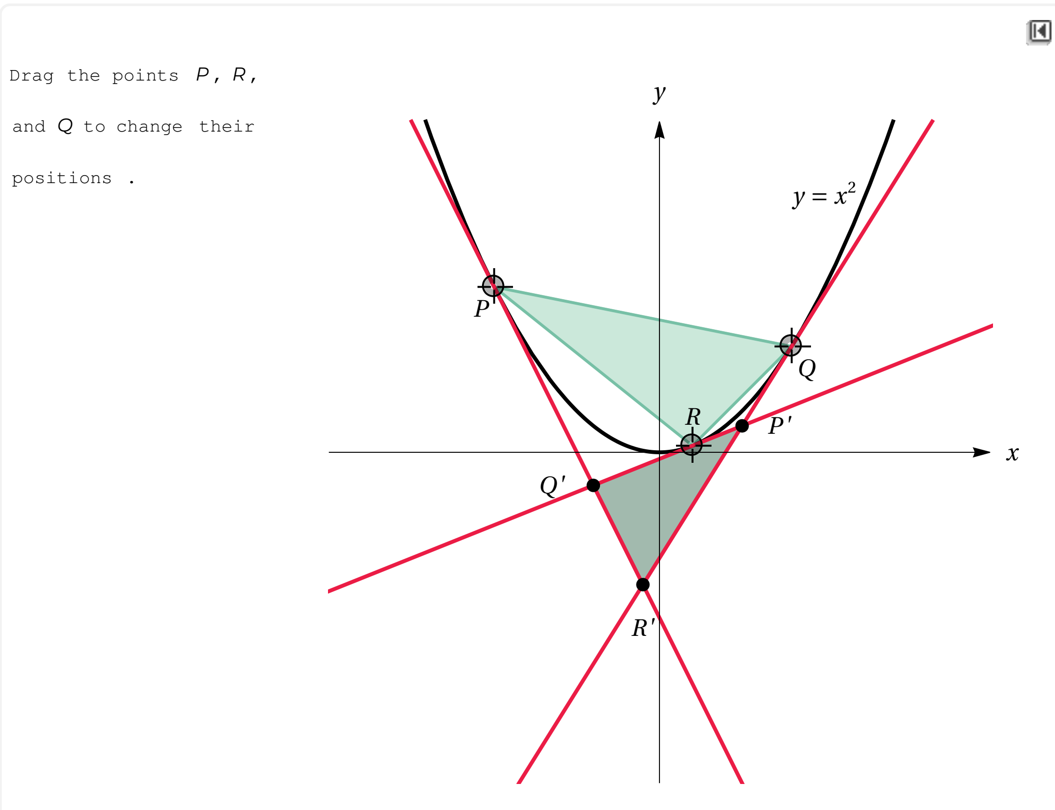
d. The information in the Lorenz curve is often summarized in a single measure called the **Gini index**, which is defined as follows. Let A be the area of the region between $y = x$ and $y = L(x)$ (see figure) and let B be the area of the region between $y = L(x)$ and the x-axis. Then the Gini index is $G = \frac{A}{A+B}$. Show that $G = 2A = 1 - 2\int_0^1 L(x)\,dx$.

e. Compute the Gini index for the cases $L(x) = x^p$ and $p = 1.1, 1.5, 2, 3,$ and 4.

f. What is the smallest interval $[a, b]$ on which values of the Gini index lie for $L(x) = x^p$ with $p \geq 1$? Which endpoints of $[a, b]$ correspond to the least and most equitable distribution of wealth?

g. Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions $L(0) = 0$, $L(1) = 1$, and $L'(x) \geq 0$ on $[0, 1]$. Find the Gini index for this function.

76. **Equal area properties for parabolas** Consider the parabola $y = x^2$. Let P, Q, and R be points on the parabola with R between P and Q on the curve. Let ℓ_p, ℓ_Q, and ℓ_R be the lines tangent to the parabola at P, Q, and R, respectively (see figure). Let P' be the intersection point of ℓ_Q and ℓ_R, let Q' be the intersection point of ℓ_P and ℓ_R, and let R' be the intersection point of ℓ_P and ℓ_Q. Prove that Area $\Delta PQR = 2 \cdot$ Area $\Delta P'Q'R'$ in the following cases. (In fact, the property holds for any three points on any parabola.) (*Source: Mathematics Magazine*, 81, 2, Apr 2008)



a. $P(-a, a^2)$, $Q(a, a^2)$, and $R(0, 0)$, where a is a positive real number

b. $P(-a, a^2)$, $Q(b, b^2)$, and $R(0, 0)$, where a and b are positive real numbers

c. $P(-a, a^2)$, $Q(b, b^2)$, and R is any point between P and Q on the curve

T **77. Roots and powers** Consider the functions $f(x) = x^n$ and $g(x) = x^{1/n}$, where $n \geq 2$ is a positive integer.

a. Graph f and g for $n = 2, 3,$ and 4, for $x \geq 0$.

b. Give a geometric interpretation of the area function $A_n(x) = \int_0^x (f(s) - g(s))\, ds$, for $n = 2, 3, 4, \ldots$ and $x > 0$.

c. Find the positive root of $A_n(x) = 0$ in terms of n. Does the root increase or decrease with n?

T **78. Shifting sines** Consider the functions $f(x) = a \sin 2x$ and $g(x) = \frac{\sin x}{a}$, where $a > 0$ is a real number.

a. Graph the two functions on the interval $\left[0, \frac{\pi}{2}\right]$, for $a = \frac{1}{2}$, 1, and 2.

b. Show that the curves have an intersection point x^* (other than $x = 0$) on $\left[0, \frac{\pi}{2}\right]$ that satisfies $\cos x^* = \frac{1}{2a^2}$, provided $a > 1/\sqrt{2}$.

c. Find the area of the region between the two curves on $[0, x^*]$ when $a = 1$.

d. Show that as $a \to 1/\sqrt{2}^{\,+}$, the area of the region between the two curves on $[0, x^*]$ approaches zero.